416

PROCEEDINGS

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From December 1914 *to August* 1915.

VOL. XXVII.

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PROCEEDINGS

AT THE

MEETINGS OF THE PHYSICAL SOCIETY

OF LONDON.

SESSION 1914–1915.

October 23, 1914.

Meeting held at the Imperial College of Science.

Prof. Sir J. J. THOMSON, O.M., F.R.S., President, in the Chair.

The following Paper was read :—

"Ionisation," Presidential Address by Sir J. J. THOMSON.

November 13, 1914.

Meeting held at the Imperial College of Science.

Dr. A. RUSSELL, M.A., Vice-President, in the Chair.

The following Papers were read :—

1. "A Bridge for the Measurement of Self-Induction." By D. OWEN, B.A., B.Sc.
2. "On the Coefficient of Diffusion in Dilute Solutions." By B. W. CLACK, B.Sc.

November 27, 1914.

Meeting held at the Imperial College of Science.

Dr. A. Russell, M.A., Vice-President, in the Chair.

The following Papers were read :—

1. "Note on the Conduction of Electricity at Point Contacts." By A. F. Hallimond.

2. "The Thermal Conductivity of Badly Conducting Solids." By T. Barratt, B.Sc.

December 18, 1914.

Meeting held at the Imperial College of Science.

Dr. A. Russell, M.A., Vice-President, in the Chair.

The following Papers were read :—

1. "An Exhibition and Description of Some Apparatus for Class Work in Practical Physics." By Dr. G. F. C. Searle, F.R.S.

2.* "A Vacuum Guard Ring and its Application to the Determination of the Thermal Conductivity of Mercury." By H. R. Nettleton, B.Sc.

January 22, 1915.

Meeting held at the Imperial College of Science.

Prof. Sir J. J. Thomson, O.M., F.R.S., President, in the Chair.

The following Papers were read :—

1. "Practical Harmonic Analysis." By Dr. A. Russell, M.A.

2. "Measuring the Focal Length of a Photographic Lens." By T. Smith, B.A.

3. "The Polyscope and its Projection." By Prof. A W. Bickerton.

* Taken as read.

Annual General Meeting.

February 12, 1915.

Meeting held at the Imperial College of Science.

Dr. A. RUSSELL, M.A., Vice-President, in the Chair.

The Report of the Council was taken as read.

In the year 1914 fourteen ordinary meetings have been held. The meeting on June 20th was noteworthy in that it was held at Cambridge. About 100 members and visitors took advantage of the occasion to visit the works of the Cambridge Scientific Instrument Co., and were subsequently the guests of the Company at a luncheon held by kind permission of the Master and Fellows in the Hall of St. John's College. In the afternoon a science meeting was held in the Cavendish Laboratory, and was followed by tea in the laboratory by the invitation of the President and Lady Thomson. The average attendance at the meetings was 56.

On February 27th the first Guthrie Lecture was delivered by Prof. R. W. Wood, of Johns Hopkins University, Baltimore, his subject being "Radiation of Gas Molecules Excited by Light," and was much appreciated by a large audience.

During the year a report on "Radiation and the Quantum Theory," by Mr. J. H. Jeans, F.R.S., was published by the Society. It is intended that this shall form the first of a series of reports on various branches of physics.

A report of the Committee on Nomenclature and Symbols was published in the August number of the "Proceedings," so that it may be discussed before the Council take any definite action. A second report will be published shortly.

The Annual Exhibition of Apparatus was not held during the past year, as the Council felt it would not be fully supported owing to the war.

The Presidential Address this year was delivered at the first autumn meeting instead of at the Annual General Meeting, as had been the case hitherto, it being felt that the autumn provided a more convenient time of year for a new President to deliver his address.

The number of Ordinary Fellows on the roll at December 31, 1914, as distinct from Honorary Fellows, was 454. Twenty-eight new Fellows have been elected, and there have been eight resignations.

The Society has to mourn the loss of one Honorary Fellow, Prof. W. Hittorf, and six Ordinary Fellows—namely, Mr. R. Kaye Gray, Prof. G. M. Minchin, Prof. J. H. Poynting (one of the Society's Past Presidents), Lieut.-Col. M. T. Sale, Mr. Augustus Stroh, and Sir Joseph W. Swan.

The Report was adopted by the Meeting.

The Report of the Treasurer and the Balance-sheet were presented by the Treasurer.

The total income of the Society again shows an improvement over the preceding year, in spite of the adverse conditions existing in the latter part. It is satisfactory to note that there is a material increase in the income derived from subscriptions and from the sales of publications. The steady increase in the sales of publications seems to indicate that the activity of the Society is being more and more appreciated by the scientific world.

The expenditure for the year has increased, due in part to the greater activity in publication, and also to the more forward policy of the Council with regard to lectures and scientific reports.

The Society's income this year has exceeded its expenditure by £101. 16s. 3d., which I think shows that the Society is in a sound financial condition.

Owing to the favourable conditions which prevailed at the beginning of 1914, the Council was able to order an investment of some of the cash assets, which stood in the last accounts at £698. 14s. I, therefore, purchased on behalf of the Society, £500 India 3½ per cent. Stock.

The total assets of the Society show a slight increase, but too much importance should not be attached to the figures given for the present values of the investments. The values given for the securities were kindly supplied by Messrs. Parr's Bank, and are based on the Stock Exchange minimum prices.

The liabilities on account of the Life Compositions Fund are a little higher, in spite of the deaths of four Life Fellows, as two new Fellows have compounded for their subscriptions. The balance available in the General Fund of the Society has slightly increased since last year.

The Report of the Treasurer was adopted.

The Treasurer in Account with the Physical Society, from January, 1914, to December 31st, 1914.

Dr.	£	*s.*	*d.*	£	*s.*	*d.*
Entrance Fees	26	5	0			
Subscriptions	522	13	6			
Extra Subscriptions	19	19	0			
Arrears	8	8	0			
Subscriptions paid in Advance	6	6	0			
Students' Subscriptions	3	3	0			
Subscriptions for "Science Abstracts" and Advanced Proofs	14	3	6			
				600	18	0
Composition Fees				52	10	0
Dividends :—						
Furness Debenture Stock	15	0	6			
Midland Railway	37	12	6			
Metropolitan Board of Works	6	11	6			
Lancaster Corporation Stock	11	5	0			
New South Wales	8	7	2			
London, Brighton & South Coast Railway	24	2	5			
Great Eastern Railway	18	15	10			
India 3½% Stock	12	6	2			
				134	1	1
Interest on deposit account				6	16	4
Sales of Publications ("The Electrician" Printing & Publishing Co.)				117	10	8
				911	16	1
Transferred from Deposit Account				300	0	0
Balance at Bank, January 1, 1914	98	14	0			
Less Four Cheques	27	7	9			
				71	6	3
				£1,283	2	4

Cr.	£	*s.*	*d.*	£	*s.*	*d.*
"Science Abstracts"				281	11	0
Printing and Distributing Publications of the Society :—						
"The Electrician" Printing & Publishing Co.—						
Proceedings	230	6	4			
Bulletin	24	1	11			
Distribution	47	0	4			
General	80	7	1			
Jeans' Report	40	1	3			
	421	16	11			
Less credit for use of type	2	5	0			
				419	11	11
Periodicals	2	11	4			
Reporting	35	0	0			
Refreshments and Attendance	15	10	0			
Petty cash	23	7	5			
Bank commission	0	6	2			
Royal Asiatic Society	2	2	0			
Guthrie Lecture honorarium	10	0	0			
Jeans' Report honorarium	20	0	0			
				108	16	11
				809	19	10
Invested in £500 India Three and a Half per cent. Stock				445	13	6
Balance at Bank, January 1, 1915	32	8	5			
Less Two Cheques	4	19	5			
				27	9	0
				£1,283	2	4

WILLIAM DUDDELL, *Honorary Treasurer.*

Audited and found correct,

THOMAS H. BLAKESLEY.
J. E. KINGSBURY.

PROPERTY ACCOUNT OF THE PHYSICAL SOCIETY, DECEMBER 31ST, 1914.

ASSETS.	£	s.	d.
Subscriptions due, Treasurer's estimate...	32	0	6
£533 Furness Railway 2 per cent. Debenture Stock	380	0	0
£1,600 Midland Railway 2½ per cent. Preference Stock	944	0	0
£200 Metropolitan Board of Works 3½ per cent. Consolidated Stock	196	0	0
£400 Lancaster Corporation 3 per cent. Redeemable Stock	304	0	0
£254. 2s. 9d. New South Wales 3½ per cent. Debenture Stock	235	0	0
£500 London, Brighton & South Coast Railway Ordinary Stock	490	0	0
£500 Great Eastern Railway 4 per cent. Debenture Inscribed Stock.....	485	0	0
£500 India 3½ per cent Stock	415	0	0
Balance at Bank	32	8	5
Ditto on deposit.....	300	0	0
Publications (Treasurer's Estimate).....	220	0	0
	£4,033	8	11

LIABILITIES.	£	s.	d.
Two Cheques.....	4	19	5
Life Compositions	2,102	0	0
Balance General Fund.....	1,926	9	6
	£4,033	8	11

WILLIAM DUDDELL, *Honorary Treasurer.*

Audited and found correct,

THOMAS H. BLAKESLEY.
J. E. KINGSBURY.

LIFE COMPOSITION FUND.

	£	s.	d.
170 Fellows paid £10 ...	1,700	0	0
3 Fellows paid £15 ...	45	0	0
5 Fellows paid £21 ...	105	0	0
8 Fellows paid £31. 10s.	252	0	0
	£2,102	0	0

NOTE.—Four Fellows who paid £10 deceased during year 1914.
Two Fellows, one at 20 and one at 30 guineas compounded.

Audited and found correct,

WILLIAM DUDDELL, *Honorary Treasurer.*

THOMAS H. BLAKESLEY.
J. E. KINGSBURY.

The Election of Officers and Council then took place, the new Council being constituted as follows :—

President.—Sir J. J. Thomson, O.M., D.Sc., F.R.S.

Vice-Presidents, who have filled the Office of President.—Prof. G. C. Foster, D.Sc., LL.D., F.R.S. ; Prof. R. B. Clifton, M.A., F.R.S. ; Prof. A. W. Reinold, C.B., M.A., F.R.S. ; Prof. Sir Arthur W. Rucker, M.A., D.Sc., F.R.S. ; Sir W. de W. Abney, R.E., K.C.B., D.C.L., F.R.S. ; Prin. Sir Oliver J. Lodge, D.Sc., LL.D., F.R.S. ; Prof. Silvanus P. Thompson, D.Sc., F.R.S. ; R. T. Glazebrook, C.B., D.Sc., F.R.S. ; Prof. J. Perry, D.Sc., F.R.S. ; C. Chree, Sc.D., LL.D., F.R.S. ; Prof. H. L. Callendar, M.A., LL.D., F.R.S. ; Prof. A. Schuster, Ph.D., Sc.D., F.R.S.

Vice-Presidents.—W. R. Cooper, M.A., B.Sc. ; A. Russell, M.A., D.Sc. ; F. E. Smith, R. S. Whipple.

Secretaries.—S. W. J. Smith, M.A., D.Sc., F.R.S. ; W. Eccles, D.Sc.

Foreign Secretary.—R. T. Glazebrook, C.B., D.Sc., F.R.S.

Treasurer.—W. Duddell, F.R.S.

Librarian.—S. W. J. Smith, M.A., D.Sc., F.R.S.

Other Members of Council.—S. D. Chalmers, M.A. ; Prof. G. W. O. Howe, M.Sc. ; Prof. J. W. Nicholson, M.A., D.Sc. ; Major W. A. J. O'Meara, C.M.G. ; C. C. Paterson ; C. E. S. Phillips, F.R.S.E. ; Prof. A. W. Porter, B.Sc., F.R.S. ; Prof. O. W. Richardson, M.A., D.Sc., F.R.S. ; Prof. the Hon. R. J. Strutt, F.R.S. ; W. E. Sumpner, D.Sc.

Prince B. Galitzin was balloted for and elected an Honorary Fellow of the Society.

The following Papers were read :—

1. "A Galvanic Cell which Reverses its Polarity when Illuminated." By A. A. Campbell Swinton.

2.* "On the Criterion of Steel Suitable for Permanent Magnets." By Prof. S. P. Thompson, F.R.S.

3.† "An Investigation of the Photographic Effect of Recoil Atoms." By A. B. Wood, M.Sc., and A. I. Steven, M.A.

* Read in abstract by the Secretary in the absence of the Author.
† Taken as read.

February 26, 1915.

Meeting held at the Imperial College of Science.

Dr. A. Russell, M.A., Vice-President, in the Chair.

The following Papers were read :—

1. "Magnetic 'Character' Figures, Antarctic and International." By Dr. C. Chree, F.R.S.
2. "The Electrification of Surfaces as Affected by Heat." By Dr. P. E. Shaw.
3. "Electromagnetic Inertia and Atomic Weight." By Prof. J. W. Nicholson, M.A., D.Sc.

March 12, 1915.

Meeting held at the Imperial College of Science.

Dr. A. Russell, M.A., Vice-President, in the Chair.

The following Papers were read :—

1. "The Estimation of High Temperatures by the Method of Colour Identity." By Clifford C. Paterson, A.M.Inst.C.E., and B. P. Dudding, A.R.C.Sc.
2. "The Unit of Candle-power in White Light." By Clifford C. Paterson and B. P. Dudding.
3.* "The Relative Losses in Dielectrics in Equivalent Electric Fields, Steady and Alternating (R.M.S.)." By G. L. Addenbrooke, M.I.E.E.

March 26, 1915.

Meeting held at University College, Gower-street, by invitation of

Profs. J. A. Fleming, F.R.S., and A. W. Porter, F.R.S.

Dr. A. Russell, M.A., Vice-President, in the Chair.

The following Papers were read :—

1. "The Change of Thermal Conductivity with Fusion." By Prof. A. W. Porter, D.Sc., F.R.S., and F. Simeon, B.Sc.

* Taken as read on account of lateness of the hour.

2. "An Instrument for the Optical Projection and Delineation of Physical Curves, such as Hysteresis, Resonance and Characteristic Curves." By Prof. J. A. FLEMING, D.Sc., F.R.S.

3. "The Stability of Some Liquid Films." By Dr. P. PHILLIPS and Mr. J. ROSE INNES, M.A., B.Sc.

Before and after the Meeting a Demonstration of the Green Flash at the Setting of an Artificial Sun was given by Prof. A. W. PORTER and E. T. PARIS.

April 23, 1915.

Meeting held at the Imperial College of Science.

Dr. A. RUSSELL, M.A., Vice-President, in the Chair.

The following Papers were read :—

1. "On the Theories of Voigt and Everett regarding the Origin of Combination Tones." By Prof. W. B. MORTON, M.A., and Miss M. DARRAGH.

2. "Experiments on Condensation Nuclei produced in Gases by Ultra-violet Light." By Miss M. SALTMARSH.

3.* "On the Self-induction of Solenoids of Appreciable Winding Depth." By S. BUTTERWORTH, M.Sc.

May 14, 1915.

Meeting held at the Imperial College of Science.

Dr. A. RUSSELL, M.A., Vice-President, in the Chair.

The following Papers were read :—

1. "Precision Resistance Measurements with Simple Apparatus." By E. H. RAYNER.

2. "Some Novel Laboratory Experiments." By F. W. JORDAN, B.Sc.

3.* "On Electrically Maintained Vibrations." By S. BUTTERWORTH, M.Sc.

* Taken as read in the absence of the Author.

May 28, 1915.

Meeting held at King's College, Strand, by invitation of Profs. O. W. Richardson, M.A., D.Sc., F.R.S., and J. W. Nicholson, M.A., D.Sc.

Dr. A. Russell, M.A., Vice-President, in the Chair.

The following Papers were read :—

1. "On Numerical Relationships between Electronic and Atomic Constants." By Dr. H. S. Allen, M.A.

2. "On a Method of Calculating the Absorption Coefficients of X-Radiation." By H. Moore, A.R.C.Sc.

Prof. O. W. Richardson showed two experiments illustrating Novel Properties of the Electron Currents from Hot Metals; Prof. E. Wilson described and exhibited the Apparatus used in his recent work on High Permeability in Iron; and Mr. T. R. Merton showed an experiment illustrating the Relative Width of Hydrogen and Neon Lines.

June 11, 1915.

Meeting held at the Imperial College of Science.

Dr. A. Russell, M.A., Vice-President, in the Chair.

The following Papers were read :—

1. "The Coefficient of Expansion of Sodium." By E. A Griffiths and E. Griffiths, M.Sc.

2. "Notes on the Calculation of Thin Objectives." By T. Smith, B.A.

3. "On Tracing Rays through an Optical System." By T. Smith, B.A.

4.* "The Accuracy of the Lens and Drop Method of Measuring Refractive Index." By H. R. Nettleton, B.Sc.

* Taken as read in the absence of the Author.

June 25, 1915.

Meeting held at the Imperial College of Science.

Dr. A. RUSSELL, M.A., Vice-President, in the Chair.

The following Papers were read :—

1. "The Conduction of Electricity through Metals." By the President, Sir J. J. THOMSON, O.M., F.R.S.

2. "On an Unbroken Alternating Current for Cable Telegraphy." By Lieut.-Col. G. O. SQUIER, Ph.D.

NOTICES TO FELLOWS.

The question of using letters to denote Fellowship of the Society has been raised from time to time. The practice of Fellows has varied in the past, and it is obviously advisable that there should be uniformity. In the circumstances, the Council has decided to sanction and adopt the letters F.P.S.L. as the official indication of Fellowship of the Society. Fellows are, therefore, asked to use no other letters for this purpose.

I. *The Production of Soft Röntgen Radiation by the Impact of Positive and Slow Cathode Rays.* By Sir J. J. Thomson, *O.M., F.R.S.**

A gap of about eight octaves separates the softest characteristic Röntgen radiation yet investigated, that of aluminium, and the shortest waves in the Schumann region, those recently discovered by Prof. Lyman; the latter have a wave-length of about 9×10^{-6} cm., the former one of $3{\cdot}6\times10^{-8}$ cm. Very little is known about any radiation of intermediate wave-length, and yet the study of such radiation is essential for the determination of the structure of the atom. By its aid we might hope to gain a knowledge of the distribution of the electrons in the atom, to determine, for example, how many rings of electrons there are in the atom and the number of electrons in each ring. We associate the K and L types of radiation with the vibrations of the two rings nearest the centre of the atom, and the visible spectrum with those of the outermost ring. By using Röntgen radiation ranging in hardness from that characteristic of aluminium to the hardest we can produce, we can detect the existence and study the properties of the two rings nearest the centre. By using the light of the visible and ultra violet region we can find out a good deal about the outermost ring, but to study the intermediate rings, and thus get even an approach to the constitution of the atom, we require radiation intermediate between Schumann and ordinary Röntgen rays, a type of radiation which has not hitherto attracted much attention.

I wish in this Paper to describe some experiments recently made in the Cavendish Laboratory on methods of producing and studying radiation of this type.

In the first place I wish to describe a new method of producing this radiation, for I find that when *positive* rays impinge against a solid, radiation of this type is produced.

The apparatus is shown in Fig. 1. The positive rays are produced in the bulb A, they pass through a tube about 2 mm. in diameter and 5 cm. long in the cathode C, and strike against

* Communicated by the Author, having been read before the Physical Society, June 20, 1914.

a platinum disc, P. B is a side tube, at the end of which is the arrangement described in my book on "Positive Rays," by which a photographic plate could be exposed to the radiation in the tube: a slit was placed in front of the plate, so as to get a definite image. The plates used were Schumann plates or plates specially prepared for me by the Paget Plate Company; these are not so sensitive to the radiation as the Schumann plates, and require a longer exposure; when, however, the radiation is strong enough they are more convenient to work with. L and M are two pairs of parallel plates 6·5 cm. long and 1 mm. apart, placed so that any radiation coming from A and striking the plate P must pass between L, while any radiation from P must pass through M before it reaches the photographic plates. These plates could be connected to a large battery of small storage cells, and a potential difference of 1,000 volts established between the plates in either pair.

When the positive rays were striking against the plate an

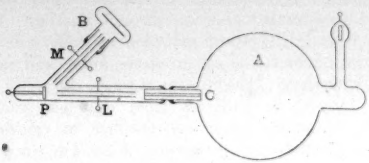


FIG. 1.

exposure of a Schumann plate for an hour gave a dense photograph of the slit, showing that something was coming down the side tube which could affect a photographic plate. There are many well recognised types of radiation produced in A, and it is necessary to make further experiments to see if the photograph is produced by these or by some new type of radiation. It might be suggested, for example, that the effect had nothing to do with positive rays passing through the cathode to P, but was produced by ultra-violet light or Röntgen rays generated in the discharge-tube, which passed through the aperture in the cathode, and was then reflected from the platinum plate. If this were the case, since neither ultra-violet light nor Röntgen rays are deflected by an electric field, the intensity of the photograph should be the same whether the P.D. between the plates in L was zero or 1,000

volts. It was found, however, that the photographic effect almost disappeared when there was a P.D. of 1,000 volts between the plates, showing that the effect is due to the positive rays. The effect, though reduced to a small fraction of its former value by putting on 1,000 volts, is yet not entirely eliminated. The small residual effect is due, I think, to the fact that the positive rays, except at the very lowest pressures, do not remain constantly positively charged, but alternate from the charged to the uncharged condition: thus a few of them might remain without charge all the time they were between the plates and thus escape deflection; after passing through the plates they might re-acquire a positive charge before striking against P. I think this is more probable than that the residual effect is due to ordinary Röntgen rays or ultra-violet light produced in A, for the rays, as we shall see, are too easily absorbed by white fluorite to be ultra-violet light, and by thin mica or collodion to be ordinary Röntgen rays. Another proof that the effect is not due to stray radiation from A is that it disappears entirely if C is made anode instead of cathode.

One of the effects of the impact of positive rays against a metal plate is to make the plate emit slow cathode rays, and it might be thought that the effect on the photographic plate was due to these rays, starting from P and travelling down to the plate. If this were the case, then putting a P.D. of 1,000 volts between the plates in M ought to stop the effect entirely. I find, however, that the photographs are just as dark when 1,000 volts are on the plates as when they are at the same potential. This seems a conclusive proof that the radiation which affects the plate is not a corpuscular radiation or a form of positive rays, but is analogous to light or Röntgen radiations—that, in fact, Röntgen radiation is produced by the impact of positive rays against a solid.

This radiation is unable to penetrate even the thinnest films I have been able to procure of substances such as collodion, mica, paraffin-wax, aluminium, or white fluorite. When part of the slit was covered with one of these films it entirely stopped the radiation through that part of the slit. This radiation can be reflected, for if a slit of the kind shown in Fig. 2 is put in front of the photographic plate, it is found that the plate is affected not only underneath A, but also on the part to the left of the opening B. Part of this reflected radiation is corpuscular, as it is affected by a magnet; a part

of it is not so affected, and so must, like the incident rays, be a form of Röntgen radiation.

I should estimate the velocity of the positive rays at about 2×10^8 cm./second; the impact of cathode rays possessing the same energy as these would generate a very hard type of Röntgen ray; the type of Röntgen ray generated by the impact of positive rays more nearly resembles that produced by cathode rays with the same velocity, but much less energy than the positive rays.

I now pass on to consider the production of soft Röntgen radiation by the impact of slow cathode rays. The arrangement used is shown in Fig. 3. C is a Wehnelt cathode—a thin strip of platinum foil with a patch of barium oxide deposited on it by burning away a speck of sealing-wax. The anode A is a piece of brass rod with a hole bored through it, through which the cathode rays pass on their way to the target B, a copper plate which is at the end of, and in metallic communication with, a cylinder of wire gauze. The variation

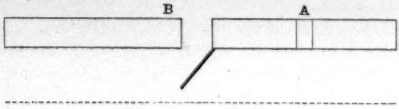


FIG. 2.

in the speed of the cathode rays was produced by putting between the gauze and the anode an E.M.F. tending to stop the rays. Thus, if V_1 is the P.D. between the anode and the cathode, V_2 that between the anode and the gauze, the energy of the cathode rays when they strike the target is proportional to V_1-V_2. This method of varying the energy of the rays was found to work better than altering the potential between the cathode and the anode, as the emission of cathode rays from C was much more regular with a constant P.D. between the anode and cathode.

To detect the radiation coming from the target, a camera similar to that used in the previous experiment was placed at the end of the side tube T. A slit was placed in front of the plate, and half of it covered by thin slices of paraffin-wax, collodion, mica, glass, or fluorite so as to estimate the penetrating power of the radiation. A magnet was placed between

the target and photographic plate so as to deflect from the latter any corpuscular radiation from the target. The vacuum was made as low as possible by means of charcoal and liquid air; it was so low that no luminosity could be detected between the anode and the target. The plates used were Schumann plates; the Paget plates were not sensitive enough to detect the radiation from the slowest cathode rays, though they gave good photographs when the rays fell through an effective potential of more than 100 volts.* The times of exposure, which varied from 1 minute to 2 hours, were chosen so as to make the energy in the cathode rays striking against the target during the time of exposure constant: thus with cathode rays which had fallen through 20 volts, the time of exposure would be 10 times that for those which had fallen

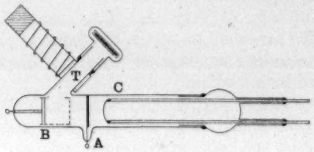


FIG. 3.

through 200; the latter gave quite dense photographs with an exposure of 2 minutes.

I have obtained photographs with cathode rays whose energy ranged from 10 to 600 volts, and there would be no difficulty in getting those corresponding to higher voltages by using larger batteries to produce the main discharge. These photographs are not due to ordinary light coming from the discharge-tube, for (i.) they are not obtained when the beam of cathode rays is deflected by a magnet from the target, and (ii.) the rays which produce them are unable to penetrate exceedingly thin films of glass To test whether they were due to a corpuscular radiation from the target two methods were employed. First, a magnet was placed between the target and the photographic plate, so as to deflect the corpuscular radiation from the plate; this did not affect the photographs. The second method was to place between the

* Since this Paper was read I have, by using a more copious supply of cathode rays, been able to get photographs at the lower voltages with Paget plates.

target and the photographic plate a pair of parallel plates similar to those used in the first experiment with positive rays, and apply to them a P.D. of 1,000 volts. The intensity of the photographs was not diminished when all the radiation which struck the plate had passed through this strong electric field, which would have stopped any charged particles.

With regard to the penetrating power of this radiation, when it is produced by cathode rays with less energy than 40 volts, I have never been able to detect any photographic effect behind a film of collodion thin enough to show the colours of thin plates, paraffin-wax 4 μ thick, mica, or thin fluorite.* When the energy corresponds to 80 volts the effect behind the paraffin and collodion is appreciable, while with 200 volts and more there is very considerable penetration of the collodion and paraffin by the rays.

The great opacity of very thin films suggests that the frequency of the rays may be within the limits of those vibrations which, according to the usual theory of dispersion, are totally reflected by a medium.

According to this theory, if the medium has only one free period of frequency n, it is impervious to light, whose frequency p is given between the limits given by the equation,

$$p=n,$$

and

$$p^2=n^2+\frac{Ne^2}{m},$$

when N is the number of electrons per unit volume, e and m are respectively the charge and mass of an electron. This relation applies when the wave-length is large compared with the distance between the molecules; and the limits of opacity depend on the degree of closeness with which the molecules are packed: for example, in a gas they depend upon the pressure. In the case of Röntgen rays when the wave-lengths are small, or even comparable with the distance between the molecules, the case is different. The effect of matter in this case is not so much to increase the refractive index as to scatter the radiation, and this scattering will be greatest when the atom is impervious to the radiation. If we apply the equation to the atom itself it indicates that the atom

* With larger currents from the Wehnelt I have been able to detect the photographic effect of the 40 volts rays behind thin collodion and mica, and also to detect the photoelectric effect and ionisation due to the rays which had passed through the films.

would be impervious to, and therefore scatter strongly, rays whose frequency is between limits which depend on the density of the electrons within the atom, and not on the closeness with which the atoms are packed.

When there are more frequencies than one intrinsic to the atom, there will be several regions of great opacity separated by intervals of comparative transparency.

I am indebted to my assistants Mr. Everett and Mr. Eagle for the assistance they have given me in these experiments.

II. *On the Measurement of the Temperature Coefficient of Young's Modulus for Metallic Wires, with Special Application to Nickel.* *By* E. PHILIP HARRISON, *Ph.D., F.R.S.E., Indian Educational Service, Professor of Physics, Presidency College, Calcutta.*

RECEIVED MAY 28, 1914.

Introduction.

THE problem of finding the variation of Young's modulus with temperature has always presented considerable difficulty, partly because small changes in temperature produce elongations in the material comparable with the stretches to be observed on loading and partly on account of the effects of the "after effect" (elastische nachwirkung) which becomes apparent at high temperatures.

To these special difficulties must be added the complexities common to the measurement of all elastic constants, such as anomalous changes in elasticity due to variations in the method of loading and annealing, or to the effect of the time during which a material has been subject to stress before its elasticity is measured. The fact remains that very little work has been published on the elastic constants of metals above 200°C., and none at all, so far as the present writer can discover, on nickel.

Of the earlier researches on the effect of temperature on elasticity the most important are those of Wertheim,* Pisati,† Katzenelsohn,‡ Macleod and Clarke.§

Pisati investigated Young's modulus for iron up to 300 deg., and expresses his results by means of a third degree empirical formula which shows a gradual diminution in the modulus as the temperature rises. Katzenelsohn experimented at lower temperatures, and Macleod and Clarke by heating a tuning fork and observing its changes in frequency deduce that the adiabatic modulus for steel decreases uniformly within the limits of their experiments. Wertheim also used an acoustical method.

* Wertheim, "Ann. Chim. Phys.," (3), Vol. XII., 1844.

† Pisati, "Gaz. Chim. Ital.," Vol. VII., p. 1; "Nuovo Cimento" (3), 4, 152, 1878; and 5, 34, 1879.

‡ Katzenelsohn, "Beiblatter," 12, p. 307, 1888.

§ Macleod and Clarke, "Phil. Trans. R.S.," Vol. CLXXI., Part 1, 1880.

Of more recent researches concerning the effect of temperature on Young's modulus only (apart from the important work of Horton and of Meissner on the rigidity modulus, or of Mallock on the bulk modulus) the following bear directly on the subject of this Paper :—

Miss Noyes * used a horizontally mounted wire, the distance between two scratches on which was determined by reading microscopes. The load was applied by adding weights to a scale pan connected to the wire by a string passing over a pulley on anti-friction rollers. Heating was effected, sometimes by a steam or hot-water jacket, sometimes by an electric current, and the experiments did not extend above 180°C.

It is claimed in the first Paper that Young's modulus varies with the temperature differently according to the method of electric heating employed. If the wire is heated by passing a current through it the modulus reaches a maximum at some temperature below 100 deg., but on heating by surrounding the wire with a helix conveying a current the modulus decreases uniformly with rise of temperature.

In the second Paper, using larger loads, the modulus was found to decrease uniformly whatever the method of electric heating. The author concludes that no detectable difference in the thermal coefficient is produced by magnetising the wire longitudinally or circularly.

Shakespear † used an interference method for measuring the stretch, and was thereby enabled to reduce the length of his specimen. The wires were heated by a steam jacket, and the temperature was measured by mercury thermometers.

He found that after a first heating the modulus was greater at higher than at lower temperatures, but that on repeated heating and cooling the material settled down to a steady state, in which Young's modulus decreased with rise of temperature. Gray Blyth and Dunlop‡ determined the variation of Young's modulus with temperature between 20 deg. and 100 deg. They used vertical wires 5 metres long, and heated with a steam jacket, reading the stretch with reading microscopes. Temperatures were measured by thermocouples. No maximum was found within the limits of their experiments, the results of which were similar to those of Shakespear.

* Noyes, "Physical Review," Vol. II., 1895, and Vol. III., 1896.

† Shakespear, "Phil. Mag.," Vol. XLVII., 1899.

‡ Gray Blyth and Dunlop, "Proc. R.S." Lond., Vol. LXVII., 1900, p. 180.

The modulus for steel at high temperatures was determined by Hopkinson and Rogers,* who heated a specimen $\frac{1}{5}$ in. in diameter and 4 in. long in an electric furnace. The experiments were carried up to 750 deg. in an atmosphere of nitrogen. Only six values of the modulus are given between 0 deg. and 750 deg., so that the authors are not in a position to express their results by means of a formula, but they find a decrease with rise of temperature and some evidence of a sudden increase in the temperature coefficient at 600 deg.

The difficulty of the experiments is increased by elastische nachwirkung, which is shown to increase with temperature. The authors claim definite values of the modulus even at 600°C. or 700°C., since they always used the "instantaneous extension" produced on first adding a load.

Wassmuth,† using a nickel wire, finds a value for the temperature coefficient up to about 100 deg. equal to $3{\cdot}26\times10^{-4}$.

A series of Papers by H. Walker‡ describe experiments on Young's modulus between 0 deg. and 180 deg. A horizontal wire method is used, similar to that employed by Miss Noyes (*loc. cit.*), whose results are to a large extent confirmed and extended to other metals. The temperature of the wires was measured from their observed resistance, the resistance temperature coefficient being known from previous experiment.

The experimental portion of the wire used by Walker was about 1 metre long. It was enclosed in a glass tube, and the scratches on the wire were observed by microscopes through two holes bored in the glass.

The main results are as follows: When a metal is heated by the ordinary method (water jacket or vapour bath) Young's modulus decreases uniformly with rise of temperature; when, however, the temperature is varied by passing a current through the wire, the resulting variation in the modulus depends on the load employed. When the latter is large the modulus changes as it does when heated ordinarily; when the load is small both magnetic and non-magnetic metals show irreversible changes, which decrease to zero as the load is increased. Moreover, for small loads the modulus reaches a maximum below 100 deg. for electric heating only.

* Hopkinson and Rogers, "Proc. R.S." Lond., Series A, Vol. LXXVI., 1905.

† Wassmuth, "Akad. Wiss. Wien Sitz. Ber.," 115, 2a, pp. 223, March, 1906.

‡ H. Walker, "Proc. R.S." Edin., 27, Part 4, No. 34, 1907; 28, Part 8, No. 40, 1908; 31, Part 1, No. 10, 1911.

Method employed in the Present Experiments.

The investigation described in this Paper was undertaken with the object of determining Young's modulus for nickel near its magnetic critical temperature, and was suggested by the fact that an anomaly in the temperature coefficient of expansion of that metal had been shown * to exist at about 370 deg. (a peculiarity subsequently investigated by Randall, and confirmed by Colby,† who used a Pulfrich interferometer). In my own experiments referred to above the wire under test had been stretched by a spring, the expansion coefficient at all temperatures being measured when the specimen was in a state of tension. It is a matter of some interest to discover to what extent, if at all, the expansion coefficient is modified by the existence of a load on the wire; one way in which this information can be obtained, apart from direct experiment, is by determining the temperature coefficient of Young's modulus and deducing therefrom the value of da/dT, where a is the linear expansion coefficient of a metal under tension T. Thus a determination of Young's modulus at different temperatures up to 500°C. would supply the required information as to the relation between stress and the expansion coefficient, and, in addition, might be expected to decide whether or not any anomalous features exist in the neighbourhood of 370 deg.

To determine the stretch for a given load with the required accuracy in a length of wire sufficiently small to ensure reasonably uniform heating is a problem to which the optical method used by Shakespear is perhaps the best adapted; but the arrangements are elaborate, even at ordinary temperatures, while above 100 deg. the experimental difficulties are materially increased. It was, therefore, decided to use the reading microscope method, which had proved entirely satisfactory in finding expansion coefficients, and to modify the apparatus so that the necessary accuracy in the measurement of the stretch could be obtained on about 20 cm. of wire.

In view of the work of Miss Noyes (*loc. cit.*) it was thought legitimate to ignore the effects, if any, of circular magnetisation; consequently the specimen was heated directly by an electric current, its temperature being determined from its resistance. By using so short a length of wire the chance of

* Harrison, "Phil. Mag.," June, 1904.
† Colby, "Phys. Review," 30, pp. 506-521, April, 1910.

" patchy " heating due to inequalities in radiating power of the metal was much reduced, while pseudo stretches caused by want of perfect straightness in the wire are also negligible. With a length of wire so great as 1 metre, such as was used by both Miss Noyes and Walker, it appears almost impossible to eliminate the chance of such sources of error in measuring the modulus, especially at temperatures below 300 deg. or 400 deg.

As regards the position of the specimen to be tested, the alternative of suspending the wire vertically and hanging weights on to its end was discarded in favour of a horizontally stretched wire, for the following reasons :—

1. Uneven heating would be less likely to occur owing to the absence of air convection currents.

2. Errors due to " sag " are negligible in so short a length as 20 cm. ; in any case such errors could be calculated and allowed for (*see* Searle, " Experimental Elasticity," p. 88).

3. The difficulty of applying the stretching force and of estimating the effect of friction when the load is not applied directly to a freely hanging wire was overcome by means of a special apparatus designed for the purpose, in which a spring under compression was used to produce the stretching force. Moreover, various associated evils, familiar to anyone who has worked on the subject, were completely eliminated, such, for instance, as sudden jerks in the stretched wire, or the difficulty of keeping the wire continually in focus when taking a series of readings over a large range of temperature.

To sum up, the method used was to observe by means of reading microscopes the elongation of about 20 cm. of the central portion of a horizontally-stretched wire, heated electrically.

The temperature was measured by observing the resistance of the wire, and the stretching force was applied by a calibrated spring in conjunction with an apparatus specially designed to avoid friction and " wobbling."

It was decided that a pair of reading microscopes (1 in. objectives) with good micrometer eye-pieces were preferable to any form of microscope moving along a mechanical slide, since the former, in the first place, afford greater accuracy in reading, and, in the second place, the fact that the microscopes themselves are fixed during a series of measurements reduces the chance of accidental displacements to a minimum. In these circumstances, if the horizontal wire method is to give reliable results, the first necessity is for the specimen of wire

to remain always in focus, notwithstanding changes in the stretching force or expansion by heat; for any readjustment of the focus is likely to alter the relative position of the microscope tubes, and thereby vitiate the results. To attain the required perfection in "focus" is was necessary to ensure that the wire when stretched or expanding should move truly parallel to itself.

Many attempts were made to apply the stretching force by means of weights hanging over systems of pulleys (Noyes and Walker), or by means of spring balances of various patterns, but in every case the results were more or less unsatisfactory. Subsequently, as above remarked, the load was applied by a spring in compression, one end of which was fixed indirectly at one end of the experimental wire and the other to the other end. (*See* Fig. 1 (*a*), which is diagrammatic, and merely shows the principle of the arrangement.) The end *a* of the wire is fastened to a rod projecting centrally from and rigidly attached to the tube T. T slides inside another tube T′, to which the right-hand end *k* of the wire is fixed; thus, if a spring, C, in compression is inserted between flanges A and B on the left-hand ends of the two tubes, the wire is subject to a tension inversely proportional to the distance AB. A large number of measurements of Young's modulus at different temperatures were made with an apparatus of this type, in which the tube T slid without appreciable "shake" inside T′, the sliding surfaces being very accurately turned and polished; but difficulties due to friction, though not so great as might be imagined, were nevertheless considerable, and the apparatus was abandoned in favour of the modification described in the next section.

Detailed Description of the Apparatus.

The apparatus finally evolved is based on the above principle, but instead of the tube T sliding inside T′ *in the ordinary way*, a system of ball bearings was introduced in a manner which will be best understood, perhaps, with the help of the diagrammatic sketches shown in Fig. 1 (*a*) and (*b*).

The tube T is provided with two flanges, XX′, one at each side, which project freely through two horizontal slots (one of which is indicated by the dotted line in Fig. 1 (*a*) in the sides of T′).

a, *b*, *c*, *d* are plates rigidly attached to T′ and parallel to the flanges. In the top and bottom surfaces of the flanges are

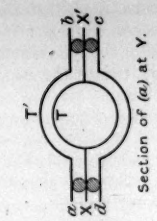


FIG. 1 (b).

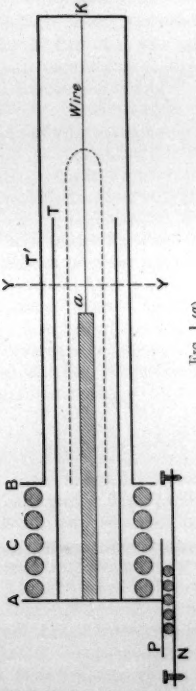


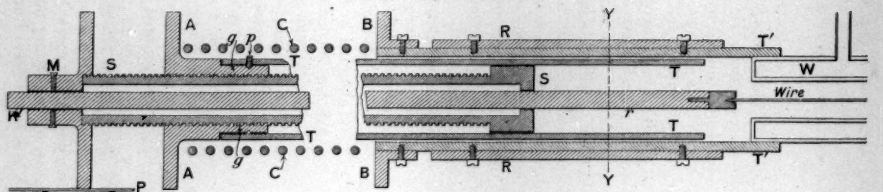
FIG. 1 (a).

machine-cut grooves parallel to the axis of the tube T, while corresponding parallel grooves run along the length of the approximating surfaces of the plates; thus between XX′ and the four plates four ball races are formed each occupied by a series of small steel bicycle balls.

The tube T is thus enabled to slide within T′, supported and guided by accurately parallel machine-cut ball bearings, with the result that the "focus" difficulty disappears, while at the same time all jerks and vibrations in the expanding wire are stopped and friction is reduced to a remarkable extent. The projecting end of T whose centre of gravity in the actual apparatus is well outside and to the left of T′ [Fig. 1 (*a*)] is supported by a plate, P, bearing by means of another parallel pair of ball bearings on a thick adjustable brass "table" N. This last detail was found to be essential to the success of the apparatus, since without it the bearings between T and T′ are apt to jam. When the instrument is in adjustment and the races well oiled the efficiency of the arrangement is very striking and, in the opinion of the writer, has rendered the horizontal wire method a most accurate and convenient one for the measurement of Young's modulus, or of an expansion coefficient.

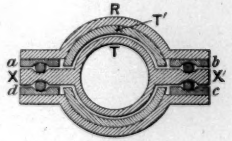
The above is a description of the principle on which the ball-bearing system was introduced. Many details have been omitted for the sake of clearness, but a drawing to scale of the complete arrangement is seen in Fig. 2 (*a*), the lettering of which applies also to the diagrammatic sketches in Fig. 1 (*a*) and (*b*). It will be seen that in practice the left-hand end of the wire is attached to a rod, *r*, which passes through a long hollow square threaded screw, *ss*. The rod *r* can be clamped at M or can slide freely without "shake" inside the screw. It is useful for purposes of adjustment. The steel compression spring C which applies the load to the wire bears on a screw nut A, working on *ss* while the nut A (and therefore the screw itself and the left-hand end of the wire) is attached to the sliding tube T. The nut is enabled, when it is screwed along *ss* to turn freely inside the tube T by means of the groove *g*, into which projects a screw, *p*; this arrangement allows A to rotate relatively to T but prevents relative motion along the axis of the tube. The reason for introducing the screw *ss* and the nut A is in order that the tension of the spring may be varied at will. It is clear that if A is screwed (say) from left to right—the right-hand end of the wire (not shown) being attached indirectly to T′—the spring is compressed and the load

Scale: Half full size.



LONGITUDINAL MEDIAN SECTION.

FIG. 2 (*a*).



CROSS SECTION AT Y.

FIG. 2 (*b*).

on the wire increased, while if A is moved in the opposite direction the load is reduced.

A water jacket, W (only a portion of which is shown in

FIG. 3 (a).

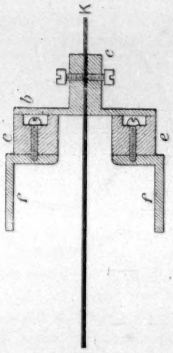


FIG. 3 (b).

Fig. 2 (*a*)), completely surrounds the wire and prevents the latter from heating up the reading microscopes and other apparatus. The right-hand end of the wire is fixed to the right-hand end of W. The simplest way to look at the somewhat complicated arrangement is to regard T, A, S and r as one system and B, T′ and W as another system capable of sliding freely relatively to one another, the two systems being tied together by the experimental wire and forced apart by the expansion of the spring C.

Fig. 3 (*a*) gives a general view of the whole apparatus, including the water jacket which surrounds the experimental wire.

The end piece E is shown in detail in Fig. 3 (*b*), and is necessary in order to insulate the two ends of the wire from one another, for the left-hand end is obviously in electric contact with the

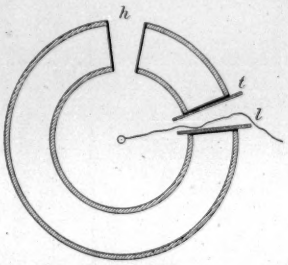


Fig. 4.

water jacket. E consists of a brass plate, *b*, carrying a clamp, *c*, separated by an ebonite block, *e*, from the brass cap *f* which screws on to the end of the water jacket. Thus, when the end piece is in position and the wire clamped at C, a heating current can be passed between K and K′.

Four holes, *h* and *h′*, *l* and *l′*, pierce the water jacket. The former pair are for viewing the scratches on the wire ; the latter are for illuminating purposes and for bringing out the potential leads used in the resistance measurement.

Fig. 4 is a cross-section of the water jacket through *h* and *l* (or *h′* and *l′*), showing the way the leads are brought out through glass tubes *t*, which do not interfere with the illumination.

The whole apparatus as shown in Fig. 3 (*a*) was mounted on the massive bed of a large comparator by Nalder Bros. with a

pair of microscopes which could be firmly clamped in any position. In order to illuminate the reference marks on the experimental wire, two straight-filament electric lamps were supported near the illumination holes l and l'. Each lamp was completely surrounded by a cylindrical water jacket pierced with a hole through which the light could fall on the aperture l or l' in the apparatus. The water circulation was found necessary to avoid heating the microscopes and the observer.

The water jacket of the main apparatus as well as those of the two lamps were in series with a coiled tube surrounding a standard manganin resistance used in connection with the resistance measurements; a continuous flow of water was kept up through the circuit during a series of observations.

Experimental Details.

General.—In making a measurement of the modulus at any temperature previous observers appear to have applied a definite load to the specimen and then to have read off the resulting elongation. The measurement was then repeated several times with the same load and the mean value of the stretches so observed was used to calculate the modulus.

The method used in the present research was to plot a load-elongation graph at each temperature, using a series of loads the greatest of which was well within the elastic limit* at that temperature. The probable error in the determination of the modulus obtained in this way is less than when a single load is applied, while additional information is supplied at each temperature as to whether Hooke's law is holding or not. The conclusion of Walker, for instance, that the elastic thermal coefficient is different for small loads from what it is for larger loads, is the same thing as stating that the load-elongation graphs are not linear over that range of temperature.

Measurement of the Stretching Force.

Calibration of the Spring.—By removing the cap E and attaching the wire, which thus passes freely out of the end of the water jacket to a specially constructed spring-balance arranged horizontally, the spring C was calibrated *in situ* under exactly the same conditions which prevail during an actual measurement of the modulus.

* *See* remarks on p. 27 as to the meaning to be attached to this expression.

The distance apart in centimetres of the flanges A and B (Fig. 1 (*a*)), was observed during the calibration.

The spring balance, in which friction was practically negligible,* had been previously calibrated in a vertical position, using 100 gramme steps.

The final calibration graph for spring C in which the distance between A and B was plotted against the tension of the spring in grammes weight proved to be a perfectly straight line within the limits of accuracy sought; this forms a good indication of the uniform nature of the extremely small frictional resistance of the ball-bearing slides.

Temperature Coefficient of the Spring.—An approximate measurement of this was made between 28°C. and 45°C. The maximum change in the temperature of the spring during one of the main experiments on Young's modulus was less than 5 deg., and it was found that the correction to be applied to the stretching force on account of changes in the elasticity of the spring was negligible.

Estimated Accuracy.—The distance AB was measured with a steel millimetre scale to $\frac{1}{10}$ mm., and, owing to slight mechanical irregularities in the flanges, was probably not correct to within $\frac{1}{5}$ mm. This means that the estimated error on a single observation of the force as read off from the calibration graph is equal to 20 grammes weight.

Measurement of the Elongation.

Considering the excellent definition of the microscopes and the fact that the wire when once mounted and annealed never got out of focus, it was found better to use small natural marks on the wire as reference marks† than to make artificial scratches. This method was not subject to accidental errors due to mistakes in identifying the marks, since any such mistakes would be detected at once when the load elongation graphs were plotted.

In all measurements the positions of the reference marks on the wire were read in terms of the "teeth" and fractions thereof in the micrometer eye-pieces; the readings were subsequently converted into centimeters by multiplying by the magnification factor.

* The spring of this balance was practically "free," the light index from its free end hardly touching the scale.

† *C.p.* Horton, "Phil. Trans." R.S., series A, Vol. ccix., p. 52.

The Magnification Factor.—This was found by placing a micrometer millimeter (by Zeiss) divided to fifths in the same plane as the surface of the wire and measuring the value of $\frac{1}{5}$ mm. in terms of "teeth" at several places along the scale. The mean value found was 1 "tooth"=0·00423 cm.

The magnification was checked by similar measurements on the divided millimeter of a standard invar meter scale.

Estimated Accuracy.—The micrometers were read to 0·01 tooth, but since the setting of the cross-wire on a mark was probably doubtful to about twice this amount, the accuracy of *an individual reading* of a reference mark is estimated at not less than 0·02 tooth, or 0·00008 cm., which is about $\frac{1}{20}$th of the degree of accuracy obtained by Shakespear in measuring the elongation by an interference method.

Measurement of the Distance between the Reference Marks.

A standard invar meter was mounted so that its scale was in the same plane as the top surface of the wire. The actual distance apart of the central fixed threads of the eye-pieces was then determined at a known temperature. This distance was found to be 20·276 cm. at 28°C. A correction on the distance between the various pairs of reference marks used at different temperatures could then be easily applied since the actual readings of the reference marks on the toothed scale of each microscope was known.

It was seldom necessary to apply this correction, since marks were usually chosen which lay near the central thread.

Estimated Accuracy.—The length measurements were correct to 0·01 cm., which was much greater than was necessary in comparison with the other measurements.

Measurement of the Area of Cross-section of the Wire.

This was found at the end of the experiments by making a series of 76 determinations of the diameter at different places along the wire with a standard screw gauge. Another series of measurements was made on a portion of the fresh unheated wire. The difference of the means of these two experiments was 0·0002 cm. The probable error of the mean of each series was much less than this, so that it was legitimate to consider the determination of the cross-section as correct to 0·000004 sq. cm. throughout the range of the experiments.

The diameter from these measurements was 0·04298 cm., giving a mean cross-section of 0·001906 sq. cm.

Estimated Accuracy of the Determination of the Radius.—The radius was taken as being correct to 0·0001 cm. throughout the range, since the error on the radius due to expansion of the wire when heated through 400 deg. was of about the same value. The correct expression $\sqrt{\left(\frac{1}{r^2}\right)_{\text{mean}}} = \frac{1}{r}$ was not employed, as the difference between this and the ordinary mean did not affect the results to 2 per cent.

Adjustment and Determination of the Temperature.

General.—The most important, and perhaps the most difficult, feature of these measurements is to keep the temperature constant during the application of a series of loads.

That this is essential is easily recognised from the following calculation :—

If $l_2 - l_1$ is the change in length of the wire for a change of temperature of θ deg., we have $l_2 - l_1 = l_1 a\theta$ approximately, where a is the average coefficient of expansion over the range θ deg. $l_1 a\theta$ is the change in length due to temperature only; consequently, in order that changes in temperature may not vitiate the measurements of elongation due to loading, we must choose θ so as to make $l_1 a\theta$ small compared with the average increment of elongation produced by a load.

Taking the elongation increment as 0·001 cm. (which is less than the average) we must make $l_1 a\theta$ less than (say) $\frac{1}{10}$th of this. That is, θ must not be greater than $0{\cdot}0001/l_1 a$.

If we take a at 250 deg. as 0·000017,* and $l_1 = 20$ cm., we get the result that θ ought not to be greater than 0·3°C. This degree of steadiness was actually attained in the large majority of the measurements. When changes in temperature materially larger than this took place the measurements were discarded.†

A battery of large capacity storage cells was used to supply the current, and the series of constantan rheostats in the heating circuit, as well as the standard manganin comparison resistance, were immersed in oil.

* Harrison, "Phil. Mag.," June, 1904.
† *See*, however, p. 34.

As has already been stated, the experimental nickel wire was completely enclosed in a water-jacket through which a circulation was kept continuously running. The wire was thus effectively protected from draughts, and prevented from heating the apparatus in its neighbourhood.

Measurement of the Resistance.

Potential leads qq' of very fine platinum were silver soldered to the wire, each about 2 cm. beyond the reference marks mm' (*see* Fig. 5). The smallest possible quantity of solder was used so as to avoid disturbing the uniformity of the heating. These two leads were brought out of the water-jacket through the illumination holes. The resistance of the portion qq' of the nickel was determined by a potentiometer method, using as comparison standard a coil of No. 12 B.W.G. managnin immersed in oil and surrounded by a water circulation coil. The potentiometer was specially constructed for these measurements, and the various resistances were adjusted so that 1 mm.

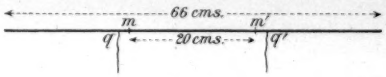


FIG. 5.

shift in the balance point corresponded to about 1°C. change of temperature in the nickel. Thus, in order to keep to the standard of temperature steadiness aimed at (0·3 deg.), observations were discarded (at least for the purpose of calculating the modulus) in which the balance point on the bridge wire changed by more than about 0·5 mm. It must be remembered, however, that such a degree of accuracy in the *measurement* of the temperature is by no means claimed. It was merely necessary for the temperature to remain constant to this extent during the observations.

A sketch of the electrical arrangements is given in Fig. 6.

Measurement of the Temperature Coefficient of Resistance of the Nickel.

The specimen of nickel wire used in these experiments was good, but was not known to be of exceptional purity. It was, therefore unjustifiable to use the results for pure nickel already

obtained by the writer in 1902.* Consequently a complete resistance temperature graph was constructed for the actual specimen employed in the Young's modulus experiments.

A suitable length of the nickel wire was wound into a spiral, fitted into a porcelain tube, and placed in an electric furnace side by side with a platinum thermometer. The two tubes were bound together with copper sheet, and their ends symmetrically placed in the centre of the furnace.

Temperature measurements were made by using the plati-

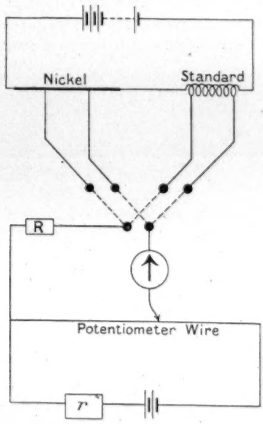


FIG. 6.

num thermometer in conjunction with a Callendar's recorder, corrections to air thermometer temperatures being applied graphically.

The resistance of the nickel at various temperatures was measured on the same potentiometer which was used in the elasticity experiments. The comparison coil was a 1-ohm standard immersed in a water bath.

The curve resulting from these resistance temperature measurements possesses a critical interval extending from

* Harrison, "Phil. Mag." S.R., Vol. III., No. 14, February, 1902.

365 deg. to 370 deg., a position which differs little from that found by the author for pure nickel (*loc. cit.* p. 24).

Estimated Accuracy.—The temperature measurements are correct to about 4°C. below 250 deg. and above 400 deg. Between 250 deg. and 400 deg. they are correct to about 3 deg. This follows from the fact that the resistance variation is more rapid near the critical temperature.

In view of the above degree of accuracy errors due to change of resistance of the wire by stretching or to changes of temperature of the standard manganin coil do not come in.

Discussion of the Accuracy of the Method.

Taking the value of Young's modulus (E) to be given by

$$E=\frac{fL}{\pi r^2 l},$$

where f is the stretching force in dynes, which produces an extension of l cm. in a wire of length L cm. and radius r cm., we have the following expressions for the proportional error ($\delta E/E$), produced in E by errors $\delta(f/l)$, δL, δr, in the quantities f/l, L and r respectively :—

Due to error in f/l—

$$\frac{(\delta E)_1}{E}=\frac{\delta f/l}{f/l}=\frac{l}{f}\delta(f/l) \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Due to error in L—

$$\frac{(\delta E)_2}{E}=\frac{\delta L}{L}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Due to error in r —

$$\frac{(\delta E)_3}{r}=-\frac{2\delta r}{r}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Now, suppose that a single observation of the stretch corresponding to a particular load is taken. It can be shown that the error on the quotient f/l, when f and l are subject to errors δf and δl respectively, is given by

$$\delta\left(\frac{f}{l}\right)=\pm\frac{\sqrt{\left(f\frac{\delta l}{l}\right)^2+(\delta f)^2}}{l},$$

where $\delta(f/l)$ is the error on a single observation of f/l.

Now, from p. 21 the value of δl for a single measurement is 0·00008 cm. and from p. 20 the value of δf for a single measure is 20 grammes weight; hence, taking $f=520$ grammes weight and the corresponding stretch $l=0{\cdot}0025$ cm. (as is the case for normal temperatures), we get

$$\delta(f/l)=\pm 10{,}000 \text{ approximately.}$$

Let it be assumed for the purposes of calculation that only five observations were made in obtaining a load elongation graph, then if each value of f/l is given the same "weight," the probable error of the mean of the five observations is

$$\frac{1}{\sqrt{5}}\delta\left(\frac{f}{l}\right)=4{,}600.$$

From expression (1) above, the percentage error produced in E by an error of 4,600 in measuring f/l is therefore given by

$$\frac{100\cdot(\delta E)_1}{E}=\frac{100\times 0{\cdot}0025\times 4{,}600}{520}=2 \text{ per cent.}$$

Again, from p. 21, the estimated error in L is 0·01 cm., so that from expression (2)

$$\frac{100(\delta E)_2}{E}=\frac{0{\cdot}01}{20}=0{\cdot}05 \text{ per cent.,}$$

and from p. 22 the estimated error in r is 0·0001 cm. Hence, from expression (3),

$$\frac{100(\delta E)_3}{E}=\frac{2\times 0{\cdot}0001}{0{\cdot}0246}=0{\cdot}8 \text{ per cent.}$$

Combining these three it appears that the error on an individual measurement of Young's modulus, apart from the influence of errors due to temperature changes, is about 2 per cent., which is all that is required considering the temperature difficulties.

The above gives an idea of the accuracy of the method at ordinary temperature when the latter does not change by more than 0·3°C. At higher temperatures the probable error in the present research is slightly less than 2 per cent., owing to the existence of larger elongations for a given load.

If the length of the experimental portion of the wire were increased, and a somewhat larger spring employed, measure-

ments of the modulus could easily be made to $\frac{1}{10}$th of 1 per cent.

Method of taking the Measurements.

The wire was annealed by heating electrically to about 500°C. under a tension sufficient to straighten out large kinks and bends in the wire. The temperature was then reduced to 300 deg., the load gradually and carefully increased at this temperature, the corresponding stretches being measured, and the process continued until no further anomalous elongations were observed (due to kink-straightening).

A load elongation graph was then determined at the temperature of the laboratory (about 27°C.).

The temperature was next raised to about 50 deg. and another graph obtained; and so on by steps of 30 deg. or 50 deg. up to 450 deg.

During the process a return was often made to lower temperatures. Throughout the range from 350 deg. to 430 deg., where anomalous features in the modulus were detected, the temperature intervals were much more frequent, and the measurements were made with continuously rising and continuously falling temperatures. As a rule, only one or two graphs could be obtained in one working day, as it took some time for the temperature to become steady, but throughout the course of the research the final form of the apparatus gave no trouble whatever, and when once in adjustment remains so indefinitely; as was emphasised before, the chief difficulties are connected with the temperature steadiness.

The total load on the wire varied from 2,000 grammes weight to 40,000 grammes weight up to a temperature of 200 deg., from 1,800 to 3,000 up to 350 deg. and from 1,600 to 2,800 up to 450 deg. The usual load increment applied was about 200 or 250 g.w. In forming a graph of load elongation, the actual load increment was plotted against the corresponding stretch in the experimental portion of the wire; up to 350 deg. a complete series of measurements included observations during unloading as well as during loading. Above this temperature the effects of elastische nachwirkung began to make themselves felt for the higher loads so that the graph for unloading was meaningless from the point of view of these experiments. In fact, all those portions of the graphs where the "after effect" became apparent correspond to what Hopkinson and Rogers call the "instantaneous extension," and this is what was employed in calculating the modulus.

When the graphs* had all been plotted, a load increment of 520 g.w. was chosen (equal to 1 cm. compression of the spring), and the corresponding stretches were read off from the graphs. The quotient of the corrected length† of the wire by the stretch gives L/l for each temperature; this is a quantity proportional to Young's modulus, and is shown in column 2, Table I.

The following table gives the data from which Fig. 7 was plotted:—

TABLE I.

Temperature of wire in degrees C.	$\frac{L}{l}$ (proportional to E).	$\frac{L}{l} \times 2{\cdot}73$ = E in kilos. per sq. mm.
27·0	8,300	22,700
27·5	8,055	22,000
96·0	7,800	21,300
121·0	7,800	21,300
145·0	7,600	20,700
185·0	7,440	20,300
222·0	7,440	20,300
236·0	7,350	20,000
255·0	7,110	19,400
285·0	7,110	19,400
329·0	6,530	17,800
350·0	6,200	16,900
360·0	5,970	16,300
376·0	5,890	16,100
382·0	5,754	15,700
396·0	5,754	15,700
397·0	5,690	15,500
401·0	5,754	15,700
406·0	5,890	16,100
409·0	5,820	15,900
422·0	5,820	15,900
425·0	5,820	15,900
430·0	5,690	15,500
432·0	5,620	15,300
437·0	5,370	14,700
448·0	4,690	12,800
454·0	4,480	12,200
465·0	4,360	11,900

Length of the wire between the marks ...	20·276 cm.
Mean radius of wire ...	0·04928 cm.
Mean cross-section ...	0·001906 sq. cm.
Load increment ...	520 grammes weight.
Total minimum load ...	From 1,600 to 2,000 grammes weight.
Value of Young's modulus at 0° C. (by extrapolation) ...	22,200 kilogrammes/sq. mm.

* Typical examples are shown in Fig. 8, *a*, *b*, *c*, *d* and *e*.
† *See* p. 21.

The multiplying factor for converting L/l into E is, of course, $\frac{f}{\pi r^2}$, and is equal to $\frac{0{\cdot}520}{0{\cdot}1907}=2{\cdot}727$ kilos. per square millimetre. The values of Young's modulus so obtained are shown in column 3, and are expressed in kilogrammes per square millimetre. They are exhibited graphically in Fig. 7.

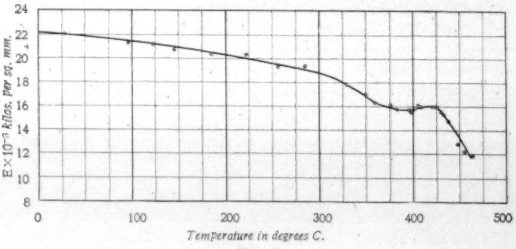


FIG. 7.

Discussion of Results.

General.—Up to about 300°C. the modulus for nickel diminishes slowly as the temperature rises. The results up to this temperature are represented well by the parabola

$$E_\theta=E_0(1-0{\cdot}0002860-0{\cdot}00000084650^2),$$

where E=22,200 kilos. per square millimetre for the specimen used. This formula was calculated for the complete series up to 325 deg. by the Method of Least Squares.

Above 325 deg. the modulus begins to decrease rather more rapidly, reaching a minimum just below 400 deg. It then appears to increase again slightly, or at least to remain nearly constant up to 425 deg., after which a rapid diminution occurs.*

Too much stress ought not to be laid on the existence of an actual *minimum* at 390 deg., but the large number of observations above 350 deg. taken on different days with temperatures both rising and falling seems to establish with certainty the existence of a stationary value between 375 deg. and 425 deg. with less certain evidence of an actual minimum value. It is

* 400 deg. and 325 deg. are the mid-points of the temperature ranges throughout which T. A. Lindsay ("Proc." R. S. Edin., 29, 1908–1909) found important evolutions of heat to occur in Ni on cooling.

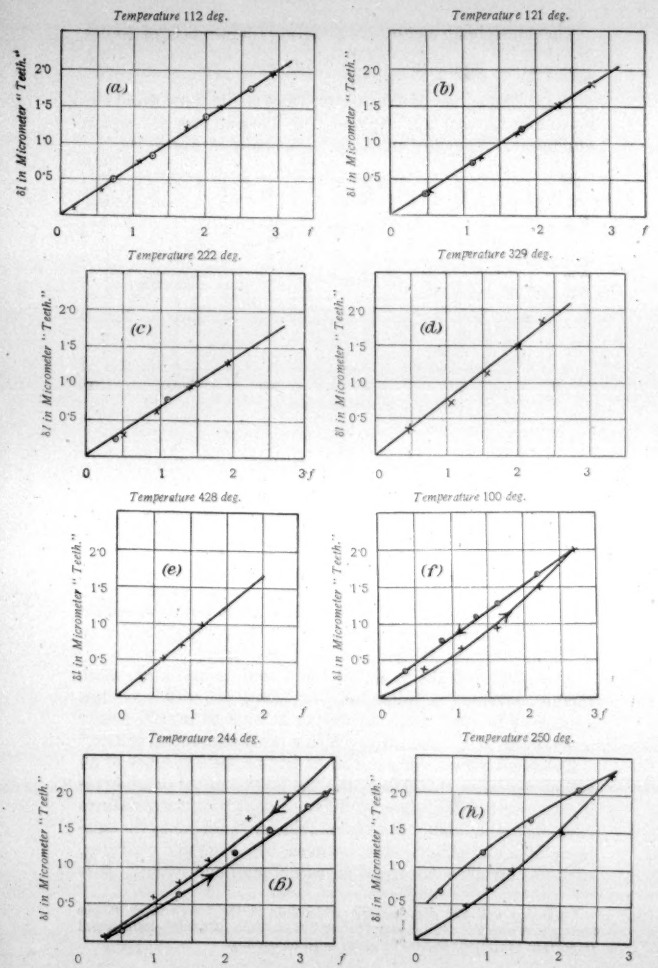

Temperature 112 deg.
(a)
Temperature 121 deg.
(b)
Temperature 222 deg.
(c)
Temperature 329 deg.
(d)
Temperature 428 deg.
(e)
Temperature 100 deg.
(f)
Temperature 244 deg.
(g)
Temperature 250 deg.
(h)
δl in Micrometer "Teeth."
f

FIG. 8.

clear that the beginning of this constant value of the modulus coincides with the beginning of the critical interval in the resistance of the metal.

The Load Elongation Graphs (Fig. 8, *a*, *b*, *c*, *d*, *e*, *f*, *g* and *h*).—In general these were linear at all temperatures (*see* the examples shown in Fig. 8, *a*, *b*, *c*, *d* and *e*), but in certain experiments (*see* Fig. 8, *f*, *g* and *h*) the relation between load and extension was not found to be linear; moreover, the curve for unloading did not coincide with that for loading.

In cases in which non-linear graphs were obtained, different types appeared on different occasions, but these invariably

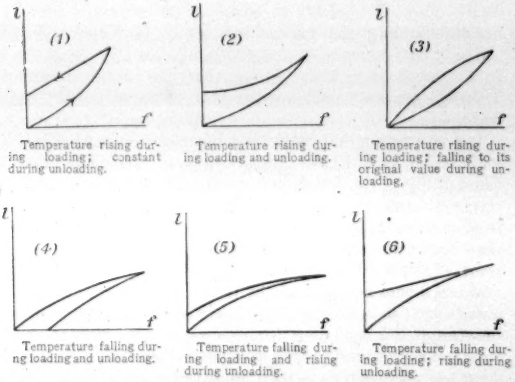


FIG. 9.

conformed to one or another of the six types shown in Fig. 9, which represent diagrammatically the obvious effect of a gradually changing temperature on an otherwise linear load elongation graph. It will be noticed that *f*, *g* and *h* are of the type shown in Fig. 9, Nos. (1), (5) and (3) respectively.

It is true that curves similar to Nos. (1) or (3) might be produced by the "after-effect" alone, provided the latter set in at some stage during loading, but in the instances referred to, not only was there no appreciable "after-effect," but the observations *when repeated under constant temperature conditions*

invariably gave a linear graph in which the values of the stretches during loading coincided with those during unloading.

The most usual form of non-linear graph obtained was like Fig. 9, Nos. (1) and (3), and in every instance of the kind it was evident from changes in the electric resistance of the wire that a slight rise, followed by a slight fall or by constancy in temperature, had occurred during the experiment.

The author, although misled at first by the regularity of such curves as *h* (Fig. 8) into a belief that they represented a genuine phenomenon connected with the elasticity of the metal, has now reached the conclusion that all the looped load elongation curves obtained in this research are due, not to an irreversible elastic effect, but merely to temperature changes. And this notwithstanding the experiments of J. O. Thomson* (who shows that increase in length is not exactly proportional to the stretching force) seeing that the effects claimed by Thomson are very much smaller than those at present under discussion.

It is interesting that Shedd and Ingersol† obtained curves for indiarubber at different temperatures of a similar kind to those in Fig. 8, No. *f*, *g*, *h*, but insufficient evidence is quoted as to the constancy of the temperature, so that in the absence of further information the present writer is not quite satisfied that Shedd and Ingersol's results cannot be explained as the result of temperature variations.

When a wire is stretched the temperature effect produced is always such as to increase the elasticity and thereby to oppose the stretch. If it be conceded, as a result of the present experiments, that the temperature coefficient of elasticity continually decreases up to 400 deg., then up to this temperature a stretch produces a fall of temperature in the wire. Hence, during loading adiabatically the temperature would be expected to fall slightly and to increase again on unloading, giving rise to the appropriate contraction and expansion in the metal. This effect, although it was actually detected in a few instances, is in general far too small to afford any explanation of the want of linearity of the load elongation graphs.

It is therefore concluded that, within the limits of accuracy discussed on pp. 21, *et seq.*, the present experiments show that the elongation of a nickel wire increases in direct proportion to

* Thomson, "Wied. Ann.," 44, 1891.
† Shedd and Ingersol, "Phys. Rev.," Vol. XIX., p. 107, 1904.

the load, provided the "instantaneous extension" is employed in all cases in which the "after-effect" is apparent. The above conclusion seems to be at variance with the results both of Miss Noyes and of Walker, who suggest that the thermal coefficient may be smaller for a small load than for a large one. Shakespear regarded effects of this nature as holding only during the first few heatings of a wire, the latter settling down to a steady state after repeated heatings and coolings, and although the present writer cannot confirm this the point is not to be regarded as settled, for if Thomson's results are accepted and hold for different temperatures some small effect of the kind claimed by Miss Noyes and by Walker may really exist. It is noteworthy that Walker quotes an incredibly small value for Young's modulus for nickel (about 14×10^{11} at 16°C.). On calculating the average thermal coefficient of the modulus from his results I find 0·06 dyne per square centimetre per degree C., which is 200 times the value given by any of the few observers who, to my knowledge, have worked on the subject

Secular Effects.

It seems likely that above 425 deg. some structural change, possibly crystalline, occurs in the metal, and that this change sets in at the beginning of the critical interval, for at 425 deg. the wire is approaching the condition in which Hooke's law ceases to be true and small loads produce a permanent stretch *which does not recover with time.*

Below 400 deg. or thereabouts, when the load exceeds a certain value the influence of "elastic after-effect" becomes increasingly apparent as the temperature rises (*c.p.* Hopkinson and Rogers), but if time for recovery is allowed the wire returns to its original length, but not to its original condition as regards elasticity. It is slightly hardened by the process. Horton noticed an analogous effect in dealing with the rigidity of metal wires (*loc. cit.*).

Consequently it appears that unless a load is sufficient at any particular temperature to induce appreciable "after-effect" no important effect is produced on Young's modulus by the resulting stress, whereas if the "after-effect" has once occurred, a higher temperature is needed to wipe out the resulting hardening.

These remarks apply to temperatures below 425 deg.; probably they do not hold above that.

If the load is so small that no "after-effect" is noticeable, it appears that the metal is softened by heating to 400 deg. or 500 deg., but that it recovers with time.

In a metal which has no critical temperature possibly the transition to the condition found in nickel above 425 deg. is gradual instead of abrupt.

Values of $\frac{d\alpha}{d\mathrm{T}}$.

It follows* from the definitions of Young's modulus (E) and the coefficient of linear expansion (α) that

$$\frac{d\alpha}{d\mathrm{T}}=\frac{d}{d\theta}\cdot\frac{1}{\mathrm{E}}=-\frac{1}{\mathrm{E}^2}\cdot\frac{d\mathrm{E}}{d\theta}, \quad \ldots\ldots \quad (4)$$

where T is the tension on the expanding wire.

Hence, obtaining the value of $\frac{d\mathrm{E}}{d\theta}$ from the formula

$$\mathrm{E}_\theta=\mathrm{E}_0(1+\mathrm{A}\theta+\mathrm{B}\theta^2),$$

and substituting in equation (4), we get

$$\frac{d\alpha}{d\mathrm{T}}=-\frac{\mathrm{E}_0}{\mathrm{E}^2}(\mathrm{A}+2\mathrm{B}\theta),$$

whence, adopting the values of A and B from the formula on p. 29,

$$\frac{d\alpha}{d\mathrm{T}}=\frac{\mathrm{E}_0}{\mathrm{E}_\theta^2}(0{\cdot}000286+0{\cdot}0000016\theta).$$

Calculating this for various values of the temperature, the following table is obtained, which, of course, applies only (strictly speaking) to the particular specimen of nickel used in this research :—

TABLE II.

θ.	E_θ in kilos. per square millimetre.	$\frac{d\alpha}{d\mathrm{T}}\times 10^6$.
10	22,180	1·37 deg.$^{-1}$kilo^{-1}mm.^{2}
100	21,840	2·09 ,,
200	20,220	3·37 ,,
300	18,650	5·08 ,,

$\mathrm{E}_0=22{,}200$ kilos./sq. mm.

* Searle, "Experimental Elasticity," p. 74, Chap. III.

The values of column 3 are shown plotted against the temperature in Fig. 10.

Expressing $\frac{da}{dT}$ as a parabolic function of θ,

we have $$\frac{da}{dT}=a+b\theta+c\theta^2.$$

Integrating this on the assumption that $\frac{da}{dT}$ is independent of T,

$$[a_\theta]_T=(a+b\theta+c\theta^2)T+P,$$

where P is a constant and is a function of θ.

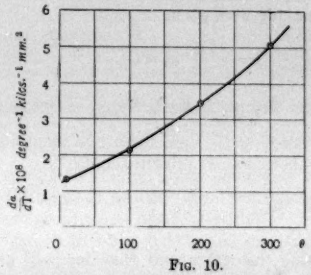


FIG. 10.

If $[a_\theta]_0$ is the value of a when T=0 we find P=$[a_\theta]_0$, so, finally

$$[a_\theta]_T-[a_\theta]_0=(a+b\theta+c\theta^2)T=T\left(\frac{da}{dT}\right)_\theta;$$

Where $\left(\frac{da}{dT}\right)_\theta$ is the value of $\frac{da}{dT}$ at temperature θ.

Since we know from Fig. 10 the variation of $\frac{da}{dT}$ with θ, it is simplest to keep the equation in the form

$$[a_\theta]_T-[a_\theta]_0=T\left(\frac{da}{dT}\right)_\theta. \quad . \quad . \quad . \quad . \quad . \quad (5)$$

and to calculate $\frac{da}{dT}$ graphically.

To express this equation in terms of the actual expansion of

the wire, let the temperature of the wire change from 0 deg. to θ deg. Then, when the tension is zero,

$$[a_\theta]_0=\frac{1}{l_0}\frac{\delta l_0}{\delta\theta}, \quad \ldots\ldots\ldots \quad (6)$$

where l_0 is the length at 0 deg. and δl_0 is the expansion for change of temperature $\delta\theta$, both measured under zero tension.

When the tension is T kilos./sq. mm.

$$[a_\theta]_T=\frac{1}{l_T}\frac{\delta l_T}{\delta\theta},$$

when l_T is the length at 0 deg., and δl_T is the expansion for change of temperature $\delta\theta$, both measured under tension T.

Now, by definition of Young's modulus,

$$E=\frac{Tl_0}{l_T-l_0} \text{ at } 0°C.,$$

or

$$l_0=l_T\left(\frac{E}{T+E}\right),$$

so the equation (6) becomes, on substituting for l_0,

$$[a_\theta]_0=\frac{T+E}{l_T E}\cdot\frac{\delta l_0}{\delta\theta},$$

whence, substituting this value and that for $[a_\theta]_T$ above, in equation (5), we have

$$\delta l_0=\frac{E}{T+E}\left[\delta l_T-l_T\,.\,\delta\theta\,.\,T\left(\frac{da}{dT}\right)_\theta\right];$$

or, since $\frac{E}{T+E}$ for tensions of the order employed is sensibly unity, we have finally

$$\delta l_T-\delta l_0=l_T\,.\,\delta\theta\,.\,T\,.\left(\frac{da}{dT}\right)_\theta. \quad \ldots\ldots \quad (7)$$

Equation (7) gives the difference between the expansion under zero load and under a load T kilos. per square millimetre, when the temperature changes from 0 deg. to $\delta\theta$.

The result is not appreciably different if we regard the temperature change $\delta\theta$ to occur at any point along the temperature scale provided l_T is then taken as being the length under tension T at the lowest temperature of the range.

Suppose, as an example, it is required to find the expansion under zero load for a nickel wire of length about 10 cm. at 100°C, when the value of the expansion under a load of 8 kilos. per square millimetre is known by experiment. We then have

$\delta\theta = 100$.

$E = 22{\cdot}2 \times 10^3$ at 0° (by present experiments).

$T = 8$ kilos. per square millimetre.

δl_T = increase in length for 100 deg. under load T.
$= 0{\cdot}014$ cm. by experiment.

$l_T = 10{\cdot}32$ cm. under tension T at 0 deg.

$\left(\frac{da}{dT}\right) = 1{\cdot}64 \times 10^{-8}$ taken from the curve Fig. 10 at 50 deg, the mean temperature of this range.

Then, from equation (7),

$$\delta l_T - \delta l_0 = 10{\cdot}32 \times 100 \times 8 \times 1{\cdot}6 \times 10^{-8} = 0{\cdot}00136 \text{ mm.}$$

Apparently the only direct experiments on the measurement of $\frac{da}{dT}$ are those of Bottomley. That observer found an extra extension of 0·14 mm. on about 500 cm. of copper wire due to a load difference of about 8 kilos. per square millimetre and a temperature difference of about 80°C.

This gives $\delta l_T - \delta l_0 = 0{\cdot}003$ mm. on 10 cm., showing that although no direct experimental data for $\frac{da}{dT}$ for nickel are as yet available, the effect obtained by Bottomley for copper is of the same order as that found theoretically for Ni from the present experiments on the temperature coefficient of Young's modulus.

Summary of Results.

1. A parabolic empirical formula is obtained which expresses the relation between Young's modulus and the temperature for nickel up to 300°C. An anomalous change is found to occur in the temperature coefficient of the modulus at the same temperature as the magnetic, thermo-electric, resistance and expansion critical points, namely, between 365 deg. and 425 deg.

2. Hopkinson and Rogers' result, that the "elastic after-

* Bottomley, "Phil. Mag.," Vol. XXVIII., 1889.

effect" in steel increases with temperature, is shown to be also true for nickel.

3. When the temperature is constant and the "elastic after-effect" is absent or allowed for, load elongation graphs are found to be linear, showing (*a*) that no appreciable irreversible effects are produced on loading and then unloading, (*b*) that the elastic temperature coefficient is the same for all loads employed during the research.

This is in conflict with the results of several other observers, as is also the fact that no maximum value in the thermal coefficient just below 100 deg. was detected, even with the smallest loads employed, which were half the value of the smallest load used by Walker. It is true, however, that the question of the existence of a maximum was not very specially investigated during the course of the present research.

4. The value of $\frac{da}{dT}$, where a is the coefficient of expansion under tension T, is deduced from the thermal coefficient of Young's modulus obtained. It can be expressed as a parabolic function of the temperature.

Hence a formula is determined which gives the difference (at any temperature) between the expansion due to heat when the wire is free and when it is subject to a load T.

I am much indebted to Dr. W. A. K. Christie, of the Geological Survey of India, for giving me the specimen of nickel with which the above experiments were carried out.

Baker Physical Laboratory,
Presidency College.

III.—*A Bridge for the Measurement of Self-Induction in Terms of Capacity and Resistance.* By DAVID OWEN, *B.A. (Cantab.), B.Sc. (Lond.), Lecturer in Physics, Birkbeck College, London.*

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1. The object of this Paper is to introduce an alternate-current bridge method of measuring inductance. This bridge is simple both in theory and practice, and proves to be adapted to accurate measurement over the whole range from a micro-henry upwards.

The use of alternating current for the accurate measurement of inductance or capacity has many advantages. The revolving commutator allows of an increase of sensibility in bridge methods where direct current is applied, but introduces a somewhat delicate and complex piece of apparatus, the use of which is obviated by applying to the bridge an alternating voltage at the outset. Where the formula of calculation is independent of frequency the method has the advantage of being unencumbered by the determination of that quantity; and, furthermore, very good work is rendered possible with no source of current more elaborate than a buzzer or small induction coil. Sensitive means of determining the attainment of a balance are available in the telephone, the vibration galvanometer and the alternating-field moving-coil galvanometer

2. *The Proposed Bridge.*—The bridge (represented in Fig. 1) serves for the determination of self-induction in terms of capacity and resistance; or, alternatively, for the measurement of capacity in terms of resistance and a standard of self-induction. We will proceed by regarding the determination as that of a self-induction L. This is placed in the second arm BC of the bridge in series with an adjustable non-inductive resistance, making up the total resistance of the arm to r_2. In the first arm AB is an accurately known non-inductive resistance r_1. The third arm AD consists of a known standard of capacity K_3, whilst the fourth arm DC contains a capacity K_4 in series with a non-inductive resistance R. On applying an alternating voltage to opposite points of the bridge no current will traverse the conjugate arm if the following conditions hold :—

$$K_3 r_1 = K_4 r_2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

$$K_3 r_1 R = K_4 r_2 R = L. \quad . \quad . \quad . \quad . \quad . \quad (2)$$

These conditions are independent of frequency, and therefore apply whatever the wave-form of the applied voltage.

The double adjustment required for balance is of the satisfactory type where the attainment of one condition remains undisturbed in the process of attaining the second. Let us suppose K_3, K_4, and r_1 to be chosen. The procedure consists in first altering r_2 until the effect in the detecting instrument is reduced to a minimum; R is then altered until the zero minimum is indicated. On repeating the process, slightly

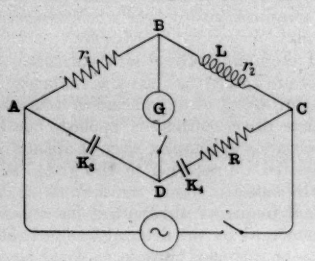


FIG. 1.—DIAGRAM OF BRIDGE.

readjusting r_2 to its best value and then R, the true balance is quickly reached. The value of the inductance required is then given by the formula $L=K_3r_1R$.

It will be observed that L is proportional to R. This leads to a result of great practical convenience, namely, the possibility of attaining a balance whatever the value of L may be. Having satisfied the first condition, one is assured beforehand that the second condition can be fulfilled. For the same reason, as will be shown later, it is possible to correct experimentally for any error due to residual inductance or capacity in the bridge.

3. *Proof of Formula.*—The formula is most readily obtained by aid of the theorem that for zero current in the galvanometer or detector the vector-impedances of branches one and two must be in the same proportion as those of branches three and four. This gives

$$\frac{r_1}{r_2+jpL}=\frac{-j/pK_3}{R-j/pK_4} \quad \ldots\ldots\ldots (3)$$

where $j=\sqrt{-1}$, and $p=2\pi\times$ frequency. Multiplying across

and equating real terms to real, and imaginary to imaginary, we find

$$\frac{r_1}{pK_4} = \frac{r_2}{pK_3} \quad \ldots\ldots\ldots \quad (1')$$

and

$$Rr_1 = \frac{L}{K_3}, \quad \ldots\ldots\ldots \quad (2')$$

which are identical with equations (1) and (2).

The bridge also lends itself to a simple graphical treatment.

In Fig. 2 ac and $a'c'$ are vectors representing the voltage at any instant between A and C. The voltages across r_1, r_2, and L are represented by ab, be, and ec respectively, the angle aec, being a right angle. Similarly the voltages across K_3, K_4 and R are represented by the vectors $a'd$, de' and $e'c'$ respectively

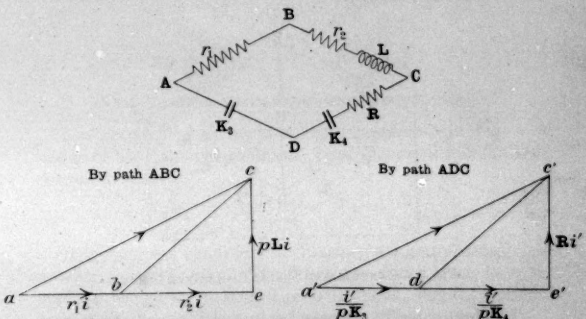


FIG. 2.—VECTORIAL REPRESENTATION OF VOLTAGES IN BRIDGE.

$a'e'c'$ being a right angle. For zero current in the galvanometer the voltage from B to D must be zero at every instant. This requires the voltages from A to B and from A to D to be equal both in magnitude and in phase; that is, the triangles ace and $a'c'e'$ must be similar and equal and b must divide ae in the same proportion as d divides $a'e'$. Denoting the current along the path ABC by i, and that along ADC by i', the above conditions are

$$\left.\begin{aligned} r_1 i &= \frac{i'}{pK_3} \\ r_2 i &= \frac{i'}{pK_4} \\ Lpi &= Ri' \end{aligned}\right\}, \quad \ldots\ldots\ldots \quad (4)$$

or
$$pK_3r_1=pK_4r_2=\frac{pL}{R}=\frac{i'}{i} \quad . \quad . \quad . \quad . \quad . \quad (4')$$

leading as before to
$$K_3r_1=K_4r_2=\frac{L}{R}.$$

We may regard the path ABC as a resistance in series with an inductance, and the path ADC as a capacity in series with a resistance. The first-named resistance is split at B, the capacity at D, in the proportions required for equal potentials at each instant at the points B and D.

4. It is of interest to consider the relation of the above bridge to a more symmetrical arrangement of inductances, resistances and capacities.

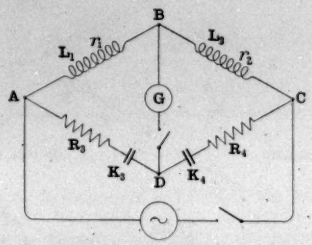


Fig. 3.—Generalised Series-Resistance Bridge.

The bridge of Fig. 3 may be described as of the "series-resistance" type. The conditions for zero current in the arm BD at each instant are

$$\left.\begin{aligned} r_1R_4-r_2R_3&=\frac{L_2}{K_3}-\frac{L_1}{K_4}\\ p^2(L_1R_4-L_2R_3)&=\frac{r_1}{K_4}-\frac{r_2}{K_3}\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (5)$$

This bridge has been employed by Rosa* and Grover for the measurement of the power-factor of a condenser, the required difference of phase angle of the two condensers being determinable in terms of the known and adjustable inductances L_1 and L_2. We can, however, eliminate the terms in p^2 by writing

* Bulletin, Bureau of Standards, Vol. III., 1907, p. 390.

$L_1=0$, $R_3=0$ (or $L_2=0$, $R_4=0$), when the above equations reduce to

$$\left.\begin{aligned} r_1R_4&=\frac{L_2}{K_3}\\ \text{and}\qquad \frac{r_1}{K_4}&=\frac{r_2}{K_3}\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

and we have a bridge adapted to the measurement of self-induction. This is the bridge described in the present Paper.

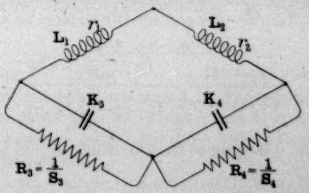


FIG. 4.—GENERALISED SHUNT-RESISTANCE BRIDGE.

If as starting point we take the arrangement in Fig. 4, a bridge of the "shunt-resistance" type, the conditions for zero current in G are

$$\left.\begin{aligned} r_2S_4-r_1S_3+p^2(K_3L_1-K_4L_2)&=0\\ (K_4r_2-K_3r_1)+(L_2S_4-L_1S_3)\quad&=0\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (7)$$

Once more the conditions for balance are complicated, and involve the frequency. The frequency terms may here be made to vanish by making K_4 and L_1 (or K_3 and L_2) equal to zero, when the conditions of balance reduce to

$$\left.\begin{aligned} r_2S_4-r_1S_3&=0\\ L_2S_4-K_3r_1&=0\end{aligned}\right\} \quad \text{or} \quad \left.\begin{aligned} \frac{r_1}{r_2}&=\frac{R_3}{R_4}\\ L_2&=K_3r_1R_4\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (8)$$

This is the Maxwell bridge. This bridge, as is well known, has the disadvantage that the two adjustments for balance specified in (8) are not independent. The modifications suggested by Rimington and by Niven have the drawback, when employed with alternating currents, of requiring a knowledge of the frequency; the formulæ, moreover, becomes rather complicated and inconvenient in use. In Anderson's well-known modification, operated with alternating current as was first done by Fleming, the two conditions of balance are made independent without sacrificing the advantage of independence of frequency.

5. *Elimination of Residual Inductance and Absorption Effects.*—It is necessary to proceed to a closer examination of the series bridge, with a view to estimating the influence of residual effects on the accuracy of the inductance determination. These residual effects are (1) the inductances l_1 of the coil r_1, l_2 of the arm r_2 (exclusive of that of the coil under test), l_3 in the leads to K_3, and l_4 in the coils R; and (2) absorption in the condensers in the arms three and four, which may be represented by resistances ρ_3 and ρ_4 in series with the capacities K_3 and K_4. In place of equation (3) we have now to write

$$\frac{r_1+jpl_1}{r_2+jp(L+l_2)}=\frac{\rho_3-\dfrac{j}{pK_3}+jpl_3}{R+\rho_4-\dfrac{j}{pK_4}+jpl_4} \quad . \quad . \quad . \quad . \quad (9)$$

leading to the following as the two conditions of balance :—

$$\frac{r_1}{K_4}=\frac{r_2}{K_3}+p^2l_1(R+\rho_4)+p^2l_4r_1-p^2(L+l_2)\rho_3-p^2l_3r_2, \quad . \quad . \quad (10)$$

and

$$K_3r_1R=L+l_2+K_3(\rho_3r_2-\rho_4r_1)-\frac{K_3}{K_4}\cdot l_1+p^2K_3l_1l_4 \\ -p^2K_3l_3(L+l_2). \quad . \quad . \quad . \quad . \quad (11)$$

The procedure for balance remains as before—namely, the adjustment of r_2, followed by that of R. One effect of the terms in p^2 is to diminish the value of r_2 to a slight extent. This is no disadvantage, since r_2 does not enter into the evaluation of L, and therefore need not be known. The second effect of the residual terms is equivalent to an addition to L, which is completely eliminated by repeating the experiment with L removed. Let R_0 denote the resistance in series with K_4 which is required for balance with L cut out, and r_2' and l_2' the new values of r_2 and l_2. Then,

$$K_3r_1R_0=l_2'+K_3(r_2'\rho_3 \quad r_1\rho_4)-\frac{K_3}{K_4}\cdot l_1+p^2K_3l_1l_4'-p^2K_3l_2'l_3 \quad (11')$$

Subtracting (11′) from (11) we have

$$K_3r_1(R-R_0)=L+(l_2-l_2')+K_3\rho_3(r_2-r_2')+p^2K_3l_1(l_4-l_4') \\ -p^2K_3l_3(L+l_2-l_2'). \quad . \quad . \quad . \quad . \quad (12)$$

Calculation shows that the only term (after the first) on the right-hand side of (12) that is of sensible value is (l_2-l_2'); so that we may write

$$K_3r_1(R-R_0)=L+(l_2-l_2'). \quad . \quad . \quad (12')$$

Now the ratio of (l_2-l_2') to L is the ratio of the time-constant of the substituted resistance coils to that of the coil under test. Taking the time-constant of the former as 10^{-8}, the error involved in neglecting (l_2-l_2') will be less than 1 in 10,000 in the case of all inductive coils whose time constant exceeds 10^{-4}; and even if coils be included for which the time-constant is as low as 10^{-5} the error is below 1 in 1,000. We may conclude that it is only rarely that it is necessary to consider the inductance of the substituted resistance coils. Our working formula thus becomes

$$\mathbf{L=K_3r_1(R-R_0)}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

Examples quoted below show that, using standard mica condensers and the usual resistance boxes, the value of R_0 is very small compared with R, except where L is very small, in which case it becomes quite important to obtain the auxiliary balance with L omitted.

It appears from the above examination that by the aid of (13) self-inductance may be determined free from error due either to residual inductance in the resistance coils used, or to absorption in the condensers.

In regard to the term (l_3-l_2') in (12′) it may be observed that the difficulty of making allowance for it is common to all bridge methods of measuring inductance.

6. *Sensitiveness of the Bridge.*—If a pure sine voltage, v, be applied between the points A and C of the bridge (Fig. 1), the current through the galvanometer arm BD is given by the expression

$$i_g=\frac{v}{A^2+B^2}\left\{A\left(\frac{L}{K_3}-r_1R\right)+B\left(\frac{r_1}{pK_4}-\frac{r_2}{pK_3}\right)-j\left[B\left(\frac{L}{K_3}-r_1R\right)-A\left(\frac{r_1}{pK_4}-\frac{r_2}{pK_3}\right)\right]\right\}, \quad . \quad (14)$$

where

$$A\equiv R(r_1g+r_1r_2+r_2g)-\frac{r_1+r_2}{p^2K_3K_4}+L\left(\frac{r_1+g}{K_4}+\frac{R+r_1+g}{K_3}\right), \quad . \quad (15)$$

and

$$B\equiv pLR(r_1+g)-\frac{pL}{p^2K_3K_4}-\frac{r_1g+r_1r_2+r_2g}{pK_3}-\frac{(g+r)(r_1+r_2)+r_1r_2}{pK_4} \quad (16)$$

The arm BD is assumed to consist of a pure resistance g; that is, inductance and back E.M.F. in the detector are neglected.

As A and B occur to the first power in the numerator of (14), but to the second power in the denominator, the current i_g for a given voltage across the bridge will be increased by reducing A and B. The expressions for A and B involve the frequency, and it appears that when $p=0$ both quantities become infinite. The bridge is thus inapplicable to steady currents. As p increases A becomes zero at a certain frequency, then increasing numerically, but with change of sign. B, like A, has the value minus infinity at zero frequency, and passes through zero at a certain frequency, generally higher than that at which A vanishes. There is, therefore, some intermediate value of the frequency for which the sensitivity of the bridge is a maximum. This value depends on the capacities employed and on the inductance to be measured. The expressions for A and B are made up of products of resistances and reactances taken three at a time, and the general rule for high sensibility is to aim at making the resistances and reactances of the arms and of the detecting instrument of approximately the same value. Using capacities each equal to $\frac{1}{3}$ m.f. it would appear that the best frequency to select in the case of inductances of the order of a tenth of a henry is about 200 per second, higher frequencies being called for the lower the inductance.

Much, however, will depend on the choice of detecting instrument. The vibration galvanometer appears to have its greatest sensitivity at low frequencies, say, 100 per second; whilst the telephone (or rather the ear) has its maximum sensitivity at 800 or 1,000$\sim$. The sensibility of the Sumpner alternate-current galvanometer is independent of frequency, provided the voltage applied to the field-coil is in proportion to the frequency. The fact that both the sensitivity of the bridge and that of the ear rise with frequency are to the advantage of the choice of the telephone, and in practice it has been found that, at all events over the lower range of inductance, the superiority as regards accuracy rests distinctly with that instrument.

From (14) it may be seen that the galvanometer current is made up of two components in quadrature; one arising from the non-attainment of the condition $L=K_2r_1R$, the other from the non-attainment of the second condition $\frac{r_1}{K_1}=\frac{r_2}{K_2}$. This result applies indeed to all bridges involving two conditions of balance.

6. *Effects of Frequency.*—Apart from the slight residual effects depending on frequency, which have already been discussed, there are three effects of frequency which may affect the value of inductance obtained by an alternating-current method. These are the skin effect (both on the inductance of the coil under test and on the resistance of the bridge coils), the effect of frequency on the capacity of a mica condenser, and, lastly, the effect arising from distributed capacity in the coil under test.

At frequencies below, say, 2,000 per second the first of them, the skin effect, is inappreciable.

The second effect is appreciable in measurements where the highest accuracy is required. A standard mica condenser shows a decrease of capacity of it may be five parts in 10,000 when the frequency rises from 100 to 500 per second. It is thus necessary that the capacity of the condenser should be known at the frequency used in the test.

The effect of distributed capacity is to cause an increase in the effective inductance of the coil tested, the increase being proportional to the value of the inductance and to the square of the frequency. The effect is only of importance in the case of large inductances and at the higher frequencies, and where necessary its value can always be experimentally determined (see, for example, observations Nos. 14 and 15 below).

7. *Experimental Arrangements and Results.*—Examples of tests are included for inductances ranging from 0·4717 henry to 2·05 microhenries. The whole range may be covered without change in the pair of condensers employed. As a rule two nominal $\frac{1}{3}$ m.f. mica standard condensers were used. The value of r_1 was 1, 10, 20, 100, or 200 ohms. To obtain the exact adjustment of r_2 and R for balance, bridge wires were inserted in the arms BC and CD. Each ot these consisted of a length of about 80 cms. of constantan wire doubled on itself, the two halves running parallel at about 0·5 cm. apart, the whole being mounted on a baseboard graduated in centimetres. A short-circuiting sliding bridge-piece allows of a continuous variation of resistance. The resistance per centimetre of wire was almost exactly 0·02 ohm. With their help exact balance could be rapidly and accurately secured, the value of R being thus determinable to within a hundredth of an ohm.

In the case of very low inductances of only a few microhenries, it is necessary to allow for the small alteration of inductance in the bridge-wire in the arm BC between the settings

for R and R_0; in other words, equation (12′) should be employed for the calculation and not (13). A continuous variation of resistance with practically constant inductance may, however, be secured by the aid of the mercury U-tube of Wenner and Dellinger, or by using a carbon plate rheostat. The latter method was used in some of the measurements recorded below, the bridge wire in arm two being dispensed with.

The *sources of current* employed were a small induction coil or buzzer giving a frequency of 95, another giving 530; a small transformer with tuning fork interrupter of 100 ∿, which provided a source of very steady frequency for use with the vibration galvanometer; also a microphonic hummer giving 512∿.

With regard to *detecting instruments*: Except in the case of the largest inductances, and of coils containing iron, the telephone in conjunction with the buzzer of 530 ∿ was found to give very satisfactory results Theoretically, owing to effects of residual inductance and of absorption, it is impossible to obtain perfect silence in the telephone unless a pure tone is employed. In practice this difficulty largely disappears. With low inductances, say, below 0·1 millihenry, the effect of overtones is almost inappreciable. At higher values it is easy after a little practice to reduce the fundamental note in the telephone to inaudibility in spite of the distinct presence of the overtones. Using a frequency as high as 530 for the fundamental tone this is particularly the case: probably only the first two, or perhaps three, overtones are effective, for not only is the intensity of the higher components present in the source smaller, but the sensibility of the ear to the corresponding tones is at the same time reduced Readings are taken of the limiting resistances at each end of the range of inaudibility—these limits are surprisingly sharp—and the mean of these readings gives the true balancing value.

For the largest inductances the vibration galvanometer was used.

For the testing of coils containing iron the telephone is practically useless. With the vibration galvanometer, on the other hand, the condition of zero movement is reached as completely as in the case of air coils, though, of course, the meaning of the resulting figures is not so easy to define.

The standard mica condenser used throughout as K_3 had the value 0·33200 m.f. at 16°C. Its value at other temperatures was read off from the curve of Fig. 5.

The condenser K_4 was a nominal $\frac{1}{3}$ m.f. mica condenser, except in experiment No. 16 (*see* footnote).

In the last column T stands for telephone, and V.G. for vibration galvanometer.

Table I.

No.	K_3(m.f.)	r_1	R	R_0	$L=K_3r_1(R-R_0)$	Estimated error (parts in 10,000).	Detecting instrument and frequency.
Single circular turn of copper wire, diameter of wire 0·071 cm., circumference of circle 150 cm.							
1	0·3318	1·0102	6·49±0·02	0·455	2·04 microhenry	64	T. 530
2	0·3320	1·008	9·72±0·02	3·50	2·08 microhenry	64	T. 530
Helix, single layer of 147 turns, length of helix 22·8 cm., external diameter 5·36 cm., diameter of wire 1·22 mm.							
3	0·33194	10	70·05±0·04	0·67	0·2303 millihenry	6·5	T. 530
4	0·33194	10	70·50±0·06	1·02	0·2306 millihenry	13	T. 95
Helix of ohmic resistance 1·1 ohm, with core of soft iron wires.							
5	0·3319	200	1539·5±0·5	0·2	0·10216 henry	6	V.G. 95
6	0·3319	200	1530·5±0·5	0·2	0·10156 henry	6	V.G. 95
Circular coil of 500 turns, mean diameter 22 cm., resistance 9 ohms.							
7	0·33194	200	1198·39±0·04	0·16	0·079548 henry	0·5	T. 530
8	0·33194	100	2397·42±0·1	0·25	0·079572 henry	0·5	T. 530
9	0·33194	200	1198·4 ±0·4	0·2	0·079540 henry	5	V.G. 95
Equal circular coils A and B, of 500 turns, mean diameter 36 cm., axial breath 2·54 cm. radial depth 5·6 cm. Placed coaxially one over the other in immediate contact.							
A only—							
10	0·33190	200	2362·19±0·4	0·2	0·15679 henry	1·7	V.G. 95
B only—							
11	0·33190	200	2403·5±0·4	0·2	0·15953 henry	1·7	V.G. 95
A and B in series, aiding—							
12	0·33190	200	7094·5±2·5	0·2	0·47090 henry	3·6	V.G. 95
A and B in series opposing—							
13	0·33190	200	2440·4±0·4	0·2	0·16199 henry	1·7	V.G. 95
From (12) and (13) we have L_A+L_B =0·31644 henry, and from (10) and (11) we have L_A+L_B =0·31632 henry; also from (12 and (13) by subtraction 4 M =0·30891 henry; ∴ M=0·07723 henry							
Coils A and B in series and aiding—							
14	0·33197	200	7104·9±1·5	0·16	0·47179 henry	2·3	T. 530
15	0·33197	200	7090·6±2·0	0·2	0·47074 henry	3·2	V.G. 95
Circular coil of 500 turns, mean diameter 22 cm., resistance 9 ohms.							
16	0·3319	200	1199·55±0·04	0·25	0·07961 henry	—	T.530

Remarks on some of the above observations :—

Nos. 1 and 2.—The inductance calculated from the dimensions of the circular wire is 2·054 microhenries (±0·005). In No. 1 a correction of +0·015 microhenry was applied to allow for the change of inductance due to movement of slider of bridge-wire in arm two [as per equation (12′)]. In No. 2 a carbon plate rheostat was used to adjust r_2, hence no such correction was required.

Nos. 5 and 6.—Although iron was present within the coil, no difficulty was experienced in obtaining a perfect balance with the vibration galvanometer. The effective resistance of the coil is at the same time ascertained ; in the case of this coil it was 49 ohms. In No. 5 the voltage applied to the bridge was 2·8 volts ; in No. 6 it was 1·7 volts.

Nos. 10 to 13.—These serve as examples of the determination of a mutual inductance.

Nos. 14 and 15.—These are observations on a large inductance, and the measurements at 530 ∼ and 95 ∼ serve for the determination of the distributed capacity. The value of L at 530 ∼ proves greater than at 95 ∼ by 20·4 parts in 10,000. Of these perhaps some 5 parts are accounted for by the diminution of K_3 with frequency. The remaining excess must be assigned to the effect of distributed capacity, as expressed in the formula

$$L_n = L_0(1+4\pi^2 n^2 k L_0),$$

where n denotes the frequency. The value of the distributed capacity k thereby calculated is $3{\cdot}9\times10^{-4}$ m.f. At the lower frequency, 95 per second, it appears that the increase in effective inductance due to distributed capacity is below 1 part in 10,000, whilst at 530 ∼ it is some 15 parts in 10,000.

No. 16.—This test differs from No. 7 in that K_4 was an ordinary commercial (Mansbridge) condenser, and not a mica condenser. Owing to the large absorption, balance with L cut out was only possible by inserting a resistance of not less than 30 ohms in series with the standard condenser K_3. The value of R_0 given above corresponds to the series resistance 30 ohms in the arm three. The value of ρ_4 deduced from the data of this balance was 5·16 ohms, K_4 being 1·838 m.f. ; and the power-factor of the condenser works out at 0·031. By comparing Nos. 7 and 16 the error in the result for L, which is introduced by the use of such a highly absorptive condenser, is seen to be less than one part in 1,000. This example shows that it is possible to obtain results sufficiently accurate for many purposes without the use of a mica condenser in the fourth arm of the bridge.

8. *The Measurement of Mutual Inductance.*—Any bridge for the determination of self- will serve for the determination of mutual-inductance. If the coils are put in series the total self-inductance is $L_1+L_2\pm2M$. Two adjustments of the bridge for balance, the second with connections of the coils reversed, lead to the difference 4M. This method is quite satisfactory where the coefficient of coupling of the coils is not too low. An example is given in Nos. 10 to 13 above.

9. *Standards of Self-Inductance.*—It is obvious that a method for the measurement of inductance in terms of capacity may be reversed. It may be regarded as a means of determining a capacity in terms of a standard of inductance. From the point of view of accuracy and reliability of results the latter may well be regarded as the proper order of procedure. An inductance standard has a very low temperature coefficient, and when

carefully designed and constructed is probably far more constant than a standard mica condenser.

Into the problem of constructing suitable primary standards of self-inductance it is not proposed to enter. It may be pointed out, however, that secondary standards are readily procurable. With the possession of such a standard the capacity of mica condensers can be checked from time to time, and their variation with temperature determined. The results of a test of temperature variation are here included. They afford, incidentally, an example of the erratic changes to which the capacity of a mica condenser is liable.*

10. *A Test of the Temperature-Coefficient of a Mica Condenser.* The measurements were made by the series-resistance bridge, the formula $K=\frac{L}{r_1(R-R_0)}$ being employed. The inductance coil was that used in tests 7 to 9 of Table I.

$r_1=200$ ohms; $R_0=0{\cdot}15$ ohm. $K_{16}=0{\cdot}33200$ m.f.

R was determined always to within 1 part in 20,000. Temperatures were ascertained to within 0·2°C.

Table II.

Temperature-variation of Capacity of Mica Condenser $v=530$.

No.....	1.	2.	3.	4.	5.	6.	7.
Temp.(°C)	18	23·2	28·5	33·2	28·3	24·2	19·3
$R-R_0$ ...	1198·37	1199·17	1200·73	1203·18	1200·68	1199·95	1199·15
Capacity	0·33212	0·33191	0·33153	0·33079	0·33148	0·33169	0·33190

No.....	8.	9.	10.	11.	12.	13.	14.
Temp.(C°)	0	10	18·7	25·0	31·0	25·6	19·4
$R-R_0$ ...	1197·25	1198·22	1199·11	1200·19	1202·49	1200·48	1199·16
Capacity	0·33244	0·33217	0·33191	0·33184	0·33163	0·33099	0·33154

From the above figures the graph of Fig. 5 is drawn. By following the numbers appended to the points on the graph it appears that between the observations 5 and 6 a sudden alteration in the capacity occurred, to the extent of 5·4 parts in 10,000. After No. 6 the results are closely expressible as a

* *See* Curtis, Bureau of Standards, Washington, November, 1910.

single-valued function of temperature. The time covered by the tests was 17 days.

The graph shows how rapidly the temperature-coefficient increases at temperatures above 15°C. Over the range

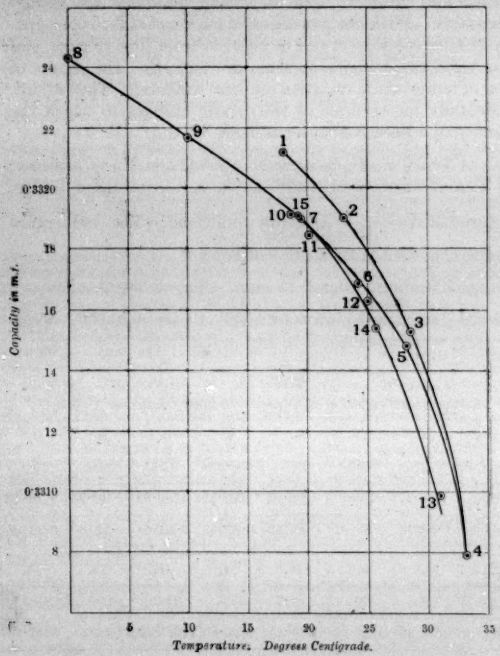


FIG. 5.—TEMPERATURE-VARIATION OF CAPACITY OF MICA CONDENSER.

0°—15°C. the mean coefficient is −0·000080, from 15° to 20°C. it is −0·00010, from 20° to 25°C. −0·00015, and from 25° to 30°C. −0·00036.

Summary.

1. An alternate-current bridge method is proposed for the determination of self-induction in terms of capacity and

resistance. The inductance L is given by the relation $L=K_3r_1R$; in addition to which it is also necessary for balance of the bridge to satisfy the condition $K_3r_1=K_4r_2$.

2. The two conditions of balance may be secured in practice without mutual interference. The end point is therefore rapidly attained. The possibility of effecting a balance is unlimited by the value of the unknown L.

3. The method is independent of frequency. This has the accompanying advantage that it is unnecessary to employ a pure sine voltage. Very good results may be attained with a buzzer as source and a telephone receiver as detector, at, say, 500 $\sim$.

4. The dependence of sensibility of the bridge on the frequency is discussed, and it is shown that over a wide range (say, 100 $\sim$ to 1,000 $\sim$) the sensibility is high.

5. The effects of residual inductance in the resistance coils and leads, and of absorption in the condensers, may be simply and correctly allowed for, the formula then becoming

$$L=K_3r_1(R-R_0).$$

This fact is of especial importance in the accurate determination of very small inductances.

6. Tests are quoted showing that with the same pair of condensers measurements over the full range from one microhenry upwards may be made. For inductances of the order of 10 microhenries the error may be kept within a few parts in 1,000; whilst if the inductance is as high as a few millihenries the error of any measurement may be reduced below one part in 10,000.

7. The application of this bridge to the determination of capacity in terms of self-inductance is discussed, and an example is given of a test of the temperature variation of capacity of a standard mica condenser over the range 0°—30°C.

ABSTRACT.

1. An alternate-current bridge method is proposed for the determination of self-induction in terms of capacity and resistance. The inductance L is given by the relation $L=K_3r_1R$; in addition to which it is also necessary for balance of the bridge to satisfy the condition $K_3r_1=K_4r_2$.

2. The two conditions of balance may be secured in practice without mutual interference. The end point is therefore rapidly attained.

The possibility of effecting a balance is unlimited by the value of the unknown L.

3. The method is independent of frequency. This has the accompanying advantage that it is unnecessary to employ a pure sine voltage. Very good results may be attained with a buzzer as source and a telephone receiver as detector, at, say, 500$\sim$.

4. The dependence of sensibility of the bridge on the frequency is discussed, and it is shown that over a wide range (say, 100$\sim$ to 1,000$\sim$) the sensibility is high.

5. The effects of residual inductance in the resistance coils and leads, and of absorption in the condensers, may be simply and correctly allowed for, the formula then becoming

$$L = K_3 r_1 (R - R_0).$$

This fact is of especial importance in the accurate determination of very small inductances.

6. Tests are quoted showing that with the same pair of condensers measurements over the full range from one microhenry upwards may be made. For inductances of the order of 10 microhenries the error may be kept within a few parts in 1,000; whilst if the inductance is as high as a few millihenries the error of any measurement may be reduced below one part in 10,000.

7. The application of this bridge to the determination of capacity in terms of self-inductance is discussed, and an example is given of a test of the temperature variation of capacity of a standard mica condenser over the range 0°—30°C.

DISCUSSION.

Mr. A. Campbell congratulated the author on his clear treatment of the general case. His method had several advantages, the most important of which was that he took a zero reading to eliminate the inductance of the leads by substituting a non-inductive resistance for the inductance to be measured. There is, of course, a difficulty in all such methods in obtaining a non-inductive resistance, but this can be got over by using a small inductance of which the value can be calculated, such as a pair of parallel wires. He could not say how far the terms in equation 12 could always be neglected as done by the author with frequencies greater than 1,000. Particular cases could be tried to see how far this was permissible. Among the disadvantages, he instanced the use of standard condensers in place of standard inductances. The latter could be obtained at less cost and of greater accuracy than condensers. Moreover, the values of condensers varied more rapidly with temperature and frequency than did those of inductances. The former defect was aggravated by the time required for a condenser to take up a steady temperature. The usual practice was to keep it at a given temperature for 24 hours or more in measuring to tenths or hundredths per cent. Another difficulty in bridge work with condensers is caused by earth capacities. If the point of earthing be altered different results may be obtained. He suggested that the galvanometer and battery might profitably be interchanged. The position described in the Paper was the worst for disturbances due to earth capacities.

Mr. F. E. Smith mentioned that in Rosa and Grover's Paper, quoted by the author, a figure practically identical with Fig. 3 in Mr. Owen's Paper appeared. Some of the equations, also, were very similar, but Rosa and Grover had, apparently, overlooked the fact that by making

one of the R's and one of the L's zero the bridge could be used for the measurement of self-induction. He was surprised that Mr. Owen appeared to prefer the telephone to the vibration galvanometer in many cases. He would have expected the latter to give more satisfaction in almost all cases. With regard to the discrepancy found by the author in the temperature curve for his condenser, he might say that in the case of paraffin wax condensers it was always dangerous to cool to 0 deg., as a change occurred in the wax which introduced trouble. He had gone into the matter of changes in the resistance of the old B.A. resistance coils when cooled to 0 deg., and had shown that these were due to changes in the wax.

Dr. RUSSELL said that he had used Mr. Owen's method of measuring inductance in his laboratory and had found it simple, accurate and most satisfactory. When ordinary alternating-current supply was available good results were obtained either with a telephone or a vibration galvanometer. The theorem on which the method is based is extremely interesting from the theoretical point of view, and several curious results follow from it. Mr. Owen's method of proof was well adapted to the needs of electrotechnical students, but he preferred the analytical methods as they were more rigorous.

The AUTHOR, in reply, referred to the variation of capacity with frequency. This was most marked at low frequencies, and as, to get sensibility, the bridge had to be used at frequencies of 100 or more, one was working outside the most dangerous region. With regard to the temperature variations, he pointed out that the condenser could always be tested against a standard inductance if need be. In the test of temperature-coefficient he had allowed the condenser to stand for four hours after the required temperature had been reached; the time to be allowed to attain the temperature of the bath could be calculated from the fact that the condenser was found to require $1\frac{1}{4}$ hours to cover half the impressed interval of temperature. In ordinary use, as the daily range was not more than 3° or 4°C., and as 1° only introduced an error of 1 in 10,000, he thought the trouble would be inappreciable. He had tried both positions of the galvanometer and battery, and sometimes one and sometimes the other proved most satisfactory. The difficulty about earth capacities was one that had to be guarded against in all balanced bridge methods when the utmost accuracy was required. The effect was likely to be very small in the bridge described, since the capacities employed were comparatively great.

IV. *On the Coefficient of Diffusion in Dilute Solutions.* By Basil W. Clack, *B.Sc., Lecturer in Physics at Birkbeck College.*

Received September 29, 1914.

§ 1. Introduction.
§ 2. Modifications in the Apparatus.
§ 3. The Apparatus.
§ 4. Results.
§ 5. Diffusivity at a definite concentration.
§ 6. The end-correction.
§ 7. Effect of the diffused salt.
§ 8. Effect of the material of the tank.
§ 9. Discussion of results.

§ 1. *Introduction.*

The author has been interested for a number of years in the investigation of the phenomena of the diffusion of salts through water, using for the purpose special flasks filled with the solution under examination, one suspended from each arm of a delicate balance, in a large bath of distilled water maintained at various constant temperatures in a thermostat room. Each flask was fitted with a vertical tube through which the salt diffused upwards and the design of the flask was such as to maintain the concentration of the solution constant at each end of the tube. The diffusion tubes of both flasks were of equal length, but their cross-sections differed considerably. A method of differential weighing was used to eliminate errors due to small changes n the density of the bath, and from the final steady rate at which the flasks changed in weight as the salt diffused out, the value of the coefficient of diffusion of the salts can be deduced.*

The objects of the present Paper are :—

1. To describe improvements made in the apparatus by means of which the results can be obtained more quickly.

2. To investigate the diffusion of certain salts down to very dilute solutions at a constant temperature.

* *See* Clack, "Proc." Ph. Soc., XXI., p. 863, 1908 ; XXIV., p. 40, 1911.

§ 2. *Modifications in the Apparatus.*

The apparatus already employed possessed the disadvantage that much time was wasted before the rate of change in weight of the flasks became approximately constant, six or seven days being required, and attempts were made to hasten the time of attainment of the steady state. In the discussion [" Proc." Ph. Soc., p. 49, 1911] Mrs. Griffiths (Miss C. H. Knowles) showed that the steady state would be hastened by filling the diffusion tubes to half their height with the solution under investigation, the rest of the tubes containing pure water. This method was tested in a number of experiments but with hardly the success expected. Not only was the steady state not accelerated very much, but the filling of the tubes seemed to produce too much disturbance, and the results obtained were not as consistent as desired. With specially careful manipulation the method might prove more successful, but in the present Paper an alternative means of hastening the steady state has been adopted, that of reducing the length of the diffusion tubes, and the arrangement at present employed is so satisfactory that the steady state is obtained almost at once. A result differing in most cases by only 1 or 2 per cent. from its final value is obtained in one day, enabling very rapid determinations of the value of the coefficient of diffusion to be made.

That shortening the diffusion tube should hasten the attainment of the steady state is well known. The case in which the liquid is moving with an impressed velocity was investigated theoretically in a note published by Miss A. Somers (" Proc." Ph. Soc., XXV., p. 74, 1912), and the present author studied the matter in 1908 in the Paper previously mentioned, by experiments on diffusion employing tubes 4 cm., 2 cm. and 1 cm. long (*see* also " Phil. Mag.," Vol. XVI., Plate XXV., Fig. 6, 1908). In this work it was observed that the end-correction to the length of the tube, which was negligible in the longest tube, because important as the length was reduced, and to avoid the error involved it has been found necessary to reduce not only the length, but also the diameter of the tubes, a battery of short and narrow tubes replacing the single wide tube previously employed. This idea was proposed by the author in 1908, in the Paper referred to, but as the same suggestion was made by Dr. Griffiths and by Prof. Lees in the discussion following the Paper read in 1911, his attention was again directed to it.

The general shape of the curves obtained by plotting the change in weight of the flasks against the time is shown in the annexed family of curves (Fig. 1) dealing with solutions of KCl of the various concentrations indicated. A comparison of

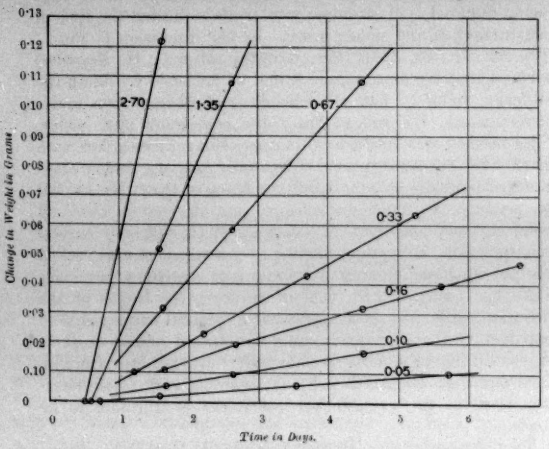


FIG. 1.

these curves with those already published will show how great has been the saving of time, an experiment which previously required 21 days, being now completed in less than a week.

§ 3. *The Apparatus.*

The general arrangement of the apparatus and the method of using it, is similar to that already described in the "Proc." Ph. Soc., December, 1911, and need not be repeated here. The aperture in each flask was closed by a glass plate drilled to receive the diffusion tubes, which were sealed in position by canada balsam.

With a battery of four parallel vertical tubes, each about 2 cm. long and 0·448 sq. cm. in area, fitted into the diffusion flask on one side of the balance and a single similar tube fitted into the other, the amount of salt diffusing was sufficient to make accurate weighings possible.

The dimensions of the tubes, found in the manner previously described, are given in the following table :—

Length.	Area.
$1{\cdot}997_8$ cm. $= L_2$	$0{\cdot}445_1$ sq. cm. $= A_2$
$2{\cdot}000_3$	$0{\cdot}454_3$
$1{\cdot}998_9$	$0{\cdot}451_3$
$2{\cdot}000_9$	$0{\cdot}440_7$
$1{\cdot}999_7$	$0{\cdot}447_4$
Mean $2{\cdot}000$ cm. $= L_1$	*Total* $= 1{\cdot}794_4$ sq. cm. $= A_1$

Results obtained with this apparatus are distinguished by the letter β in the following tables.

§ 4. *Results.*

The results for solutions of potassium chloride, potassium nitrate and sodium chloride at constant temperatures, all in the near neighbourhood of 19°C., are tabulated below. For purposes of comparison some results have been included, obtained by means of the apparatus described in 1911. These are distinguished by the letter α.

The value of the diffusivity in C.G.S. units is obtained from the formula proved in the earlier Paper.

$$K = \frac{L_1 L_2 (i_1 - i_2)}{(A_1 L_2 - A_2 L_1) N \{1 - \frac{1}{2} N \delta (D - N)\} \{1 - \delta (D - N)\}} \quad . \quad (i.)$$

where L_1 and L_2 = lengths of the diffusion tubes fitted into the two flasks.

A_1 and A_2 = their total areas of cross-section.

N = concentration of the solution in gms. per cc. of the solution at the lower end of the tubes.

D = density of the solution at the same point.

i_1 and i_2 = final steady change in weight of the two diffusion flasks per second. The balance records $(i_1 - i_2)$.

δ = ratio of the increase in volume produced to the increase in mass of salt dissolved, for a solution of concentration N.

Results to which no number is attached in the last column of the table are obtained by interpolation from the results published in the earlier Papers of the author.

KCl.

Normality.	N in gms. per cc.	Temp. in °C.	Mean slope $(i_1-i_2)\times 10^7$.	$K\times 10^5$.	Apparatus.	Experiment No.
2·70	0·2	18·5	...	1·517	α	...
		18·8	5·302	1·526	α	28
		18·7	12·38	1·550	β	10
		18·8	12·26	1·558	β	11
1·35	0·1	18·5	...	1·538	α	...
		18·6	6·124	1·496	β	8
		18·7	6·156	1·495	β	9
0·67	0·05	18·5	1·417	1·524	α	7
		18·4	1·325	1·436	α	15
		18·6	3·035	1·457	β	6
		18·9	3·130	1·473	β	12
		18·7	3·091	1·475	β	14
0·33	0·025	18·6	1·581	1·464	β	16
		18·7	1·564	1·420	β	17
0·165	0·0125	18·5	0·787	1·458	β	18
		18·6	0·745	1·380	β	19
		18·7	0·783	1·450	β	29
0·10	0·0075	18·3	0·447	1·360	β	20
		18·7	0·448	1·363	β	21
0·05	0·0037	18·3	0·228	1·366	β	22
		18·1	0·234	1·405	β	23
0·03	0·0020	17·9	0·133	1·304	β	24
		18·4	0·136	1·334	β	25

KNO_3.

1·0	0·1	18·5	...	1·319	α	...
		18·8	5·289	1·290	β	30
		19·0	5·027	1·316	β	31
0·5	0·05	18·5	...	1·333	α	...
		18·3	2·784	1·340	β	42
		19·1	2·930	1·411	β	32
		19·2	2·872	1·388	β	33
0·25	0·025	19·2	1·479	1·399	β	34
		19·2	1·555	1·408	β	35
0·10	0·010	19·1	0·620	1·454	β	36
		19·2	0·622	1·458	β	37
0·05	0·005	19·0	0·343	1·547	β	38
		18·8	0·332	1·513	β	39
0·02	0·002	18·6	0·135	1·489	β	40
		18·5	0·131	1·454	β	41

NaCl.

Normality.	N in gms. per cc.	Temp. in °C.	Mean slope $(i_1-i_2)\times 10^7$.	$K\times 10^5$.	Apparatus.	Experiment No.
3·40	0·2	18·5	10·10	1·228	β	45
		18·4	10·17	1·233	β	46
1·72	0·1	18·7	5·290	1·215	β	47
		18·7	5·213	1·198	β	48
0·85	0·05	18·5	2·977	1·200	β	49
		18·6	2·738	1·175	β	50
0·34	0·02	19·0	1·077	1·148	β	51
		18·9	1·109	1·193	β	52
0·034	0·002	19·0	0·118	1·164	β	53

The value of the coefficient of diffusion varies to some extent with the concentration of the solution, and the results here

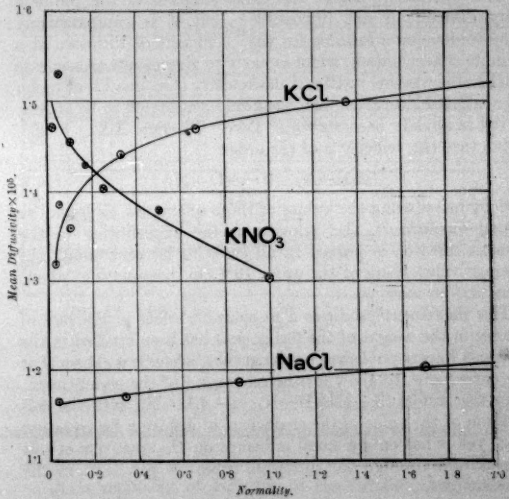


FIG. 2.

calculated give the value of what may be called the "mean diffusivity" for the solution in the diffusion tube, that is for a solution varying in concentration from zero to a maximum of

N in each experiment. The results are shown graphically in the accompanying curves (Fig. 2), in which the value of the mean diffusivity, K, is plotted against the maximum concentration, N, of the solution employed in each of the experiments.

§ 5. *Diffusivity at a Definite Concentration.*

A difficulty is encountered in attempting to find the exact value of the coefficient of diffusion at a *definite concentration* from the observations recorded in this Paper, due to the velocity with which the solution is moving down the tube. This velocity is caused by the change in volume of the solution as it becomes less concentrated by diffusion; it not only varies with the strength of the solution employed in the separate experiments, but it differs slightly at different points in the diffusion tube in the same experiment.

This movement makes the exact solution of the problem very difficult, if not impossible; but it is comparatively easy to develop a formula for the coefficient of diffusion at a definite concentration, which is likely to give results as accurate as the observations justify, if the velocity is so small that it can be neglected altogether, and the liquid considered to be at rest.

It has already been shown ("Proc." Ph. Soc. XXI., p. 863, 1908) that the velocity is of the order

$$\{i\delta(D-N)\} / \{1-\delta(D-N)\},$$

and by introducing the values of these quantities, as found by actual experiment, the velocity in the experiments of the present research is proved in all cases to be very small, the average value being of the order 10^{-7} cm. per sec., *i.e.*, about 1 cm. in four months.

This movement produces a measurable effect in the rate of change in the weight of the flasks, and has been studied in the previous Papers published by the author, where it is shown that the corrections, to allow for the movement of the water, introduce the factors $\{1-\frac{1}{2}N\delta(D-N)\}\{1-\delta(D-N)\}$ into the calculation of the value of the coefficient of diffusion by equation (i.); yet it has only a small influence on the quantity of salt transmitted through the diffusion tubes—4 per cent. in the extreme case.

Since the effect of the velocity of the water has been taken into account in arriving at Equation (i.), it seems certain that the value of the coefficient of diffusion between given limits of concentration, as calculated by this equation from the experi-

ments described in this Paper, cannot differ seriously from that value which would be obtained between the same limits of concentration were it possible to work with a perfectly stationary column of a salt solution.

Consider, then, a diffusion tube of unit cross-section so arranged that its upper end is maintained in contact with pure water and that the lower end is continually in contact with salt solution of a constant concentration, and assume that no move-

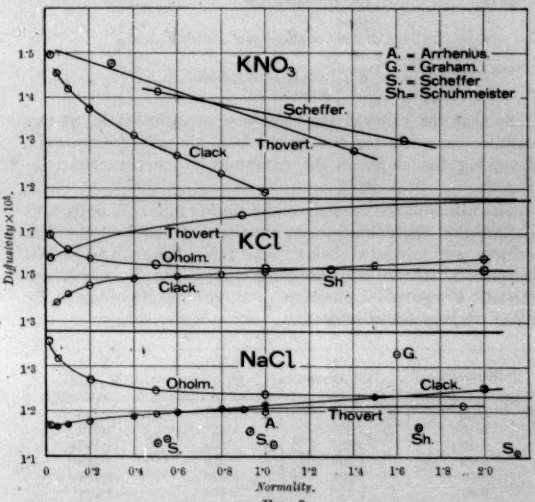


FIG. 3.

ment takes place in the water in the tube. When diffusion has reached the steady state let n be the concentration of the solution in the tube at a distance l cm. from its upper end.

Let C = constant quantity of salt transmitted per second.

${}_0K_n$ = coefficient of diffusion of the solution, varying in concentration between the limits zero and n. which satisfies the equation $C = {}_0K_n n/l$.

K_n = actual value of the coefficient of diffusion of the solution at the concentration n.

As already explained, it is assumed that the value of ${}_0K_n$ may be taken with sufficient accuracy as that obtained experimentally in this research.

Then $${}_0K_n\, n/l = K_n dn/dl = C,$$

i.e., $$K_n = C dl/dn.$$

Now $$Cl = {}_0K_n \times n$$

$$\therefore \quad C dl/dn = {}_0K_n + n \times d/dn({}_0K_n)$$

i.e. $$K_n = {}_0K_n + n \times d/dn({}_0K_n) \quad . \quad . \quad . \quad \text{(ii.)}$$

So that the value of the coefficient of diffusion K_n at any definite concentration n is found from the above expression, involving the slope of the experimental curves shown in Fig. 2.

The values of the coefficient of diffusion in C.G.S. units thus obtained at the definite concentrations indicated in the first column are tabulated below, and these figures are plotted graphically in the accompanying curves (Fig. 3), which also include, for purposes of comparison, the results obtained by other workers in this subject.

Normality.	KCl. $K_n \times 10^5$	KNO_3. $K_n \times 10^5$	NaCl. $K_n \times 10^5$
0·05	1·388	1·453	1·165
0·10	1·430	1·421	1·170
0·20	1·467	1·374	1·177
0·40	1·493	1·319	1·188
0·60	1·504	1·276	1·198
0·80	1·515	1·233	1·208
1·00	1·527	1·190	1·216
1·50	1·555	...	1·235
2·00	1·584	...	1·253

§ 6. *The End-Correction.*

In order to test whether the correction to the length of the tubes in the apparatus β was small enough to be negligible, a third pair of diffusion flasks was made, in one of which were placed 15 parallel tubes, longer and thinner than hitherto employed, and in the other was sealed a single similar tube. These tubes were each roughly 5 cm. long and 0·08 sq. cm. in

cross-section. The more exact dimensions of the various tubes are given below :—

Apparatus γ.

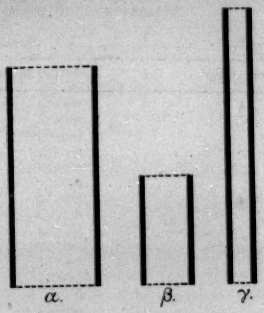
Length.	Area.
4·954 cm. $= L_2$	0·0750 sq. cm. $= A_2$
4·952	0·0859
4·954	0·0816
4·950	0·0802
4·950	0·0873
4·953	0·0810
4·949	0·0878
4·951	0·0874
4·954	0·0766
4·952	0·0744
4·952	0·0827
4·954	0·0771
4·954	0·0875
4·952	0·0807
4·951	0·0849
4·952	0·0749
Mean length 4·952 cm. $= L_1$	1·2300 sq. cm. total area $= A_1$

With this apparatus the steady state is not reached until 10 days after commencing an experiment. The results obtained for the various solutions indicated are included in the following table :—

Apparatus.	20 per cent. KCl.		5 per cent. KCl.		2 per cent. NaCl.	
	$K \times 10^5$	Expt. No.	$K \times 10^5$	Expt. No.	$K \times 10^5$	Expt. No.
α.....	1·526	28	1·480	7, 15	...	...
β	1·554	10, 11	1·473	6, 12, 14	1·170	51, 52
γ.....	1·550	43	1·462	44	1·184	54

The agreement exhibited in the results shown above, using the same solution in *different* apparatus, is as good as that obtained when the *same* apparatus is employed in different experiments, as will be seen by referring to the tables in § 4, and this is evidence that any end-corrections in the tubes used in this investigation are negligible. The relative dimensions of the diffusion tubes used in the three sets of apparatus are shown in the annexed diagram, and it will readily be granted that experiments made with such tubes should furnish a good test of the point under examination. Moreover, the concentration-gradient is reduced $2\frac{1}{2}$ times between β and γ, so that

it is clear that the coefficient of diffusion is independent of the concentration-gradient.



§ 7. *Effect of Diffused Salt.*

It is desirable to demonstrate that the contamination of the bath of distilled water by the salt which diffuses from the flasks has no appreciable effect in retarding the rate of diffusion. This was tested in an experiment by replacing the distilled water by a weak solution (0·1 per cent.) of the salt employed.

Although so dilute, this solution is much stronger than would result from the diffusion in an ordinary experiment, and yet no greater difference was found in the result than is obtained when the experiments are conducted in the usual way, provided that, in all cases, the quantity introduced into the calculation of the coefficient of diffusion is the *difference* in the concentration of the solution at the two ends of the diffusion tube. The results are tabulated below :—

Expt. No.	Concentration.	Temp. in °C.	K × 10^5.	Remarks.
10	20 p.c. KCl.	18·7	1·550	Pure water in bath.
11	,,	18·8	1·558	
27	,,	18·6	1·541	
26	20·1 p.c. KCl.	18·4	1·544	0·1 p.c. KCl in bath

§ 8. *Effect of Material of Tank.*

In order to test whether the material of which the tank is made affected in any way the rate of diffusion, the original tank which contained the bath of distilled water, made of

tinned copper, was replaced by a large glass accumulator tank of about 30 litres capacity. The result of the experiment No. 27 is included in the previous table, and demonstrates that no effect greater than that due to experimental error is caused by the tank employed.

§ 9. *Discussion of Results.*

The general conclusions to be drawn from the results of the research are that (i.) in solutions of KNO_3 the diffusivity increases as the solution becomes more dilute ; (ii.) in the case of KCl the opposite effect occurs ; (iii.) with NaCl the value of the coefficient of diffusion decreases but slightly as the concentration becomes smaller.

In all, 54 experiments have been carried out in the investigation described in this Paper. The results of the first five have been omitted, as it was in these that the method was tested of filling the diffusion tube to half its height with the solution under examination, over which pure water was placed. It has already been mentioned that this method was discontinued on account of the inconsistencies found to be associated with it. Of the remaining 49 experiments in which this method of filling was not employed, all have yielded satisfactory results, and none has been omitted, with the exception of No. 13, in which a sudden and unusually large rise of temperature in the middle of July, 1912, caused such a disturbance that no steady state was reached.

Inspection of the graphs (Fig. 3) shows a considerable difference, not only in the absolute values of the diffusivity found by different observers, using different methods, but also in the general shape of the curves. As an example in the case of KCl, Oholm (" Zeit. für Phys. Chem.," 50, p. 309, 1904-5) obtains a curve indicating that the coefficient increases with the dilution, but Thovert (" Com. Rend.," 133, p. 1197, 1901 and 134, p. 594, 1902) obtains the opposite effect. Moreover, Oholm's results are generally lower than Thovert's. It becomes necessary to consider the probable accuracy in the various methods used by experimenters in diffusion. Oholm used what may be called a " burette method," in which various specimens of the solution were run off and analysed chemically. This is a modification of one of Graham's methods (" Phil. Trans.," 183, 1861) and is a favourite, having been used by Scheffer (" Ber. Deut. Chem. Gesell. " 16, 1903, 1883 ; " Zeit. für Phys. Chem.," 2, 390, 1888), by Euler (" Wied Ann."

63, 273, 1897) and by Von Hevesey ("Phil. Mag.," 25, 390 1913).

According to the author's experience, such a method cannot be expected to yield very accurate results. A disturbance much smaller than that involved in withdrawing the specimen layers of solution for analysis causes quite fatal consequences.

On the other hand, Thovert used an interference method of determining the distribution of concentration in the diffusion cell. Such a method may be quite free from mechanical disturbance, but small temperature changes and consequent convection currents would be liable to produce severe disturbance of the displacement of the interference fringes.

Thovert measured this displacement at different levels in the diffusion cell and at two different times, so that his method is not one in which the steady state is employed.

In the present method the disturbance is quite negligible; it is only during the few minutes necessary to complete a weighing that there is any disturbance at all, and even then the movement of the suspended system is limited to $\frac{1}{10}$ mm. Moreover, the method possesses the advantage of employing the steady state. The results as indicated by the graphs are in distinct disagreement with those of Oholm, and follow the general shape of Thovert's conclusions. The agreement in the case of NaCl is very satisfactory, especially at the lower concentrations, but in the case of the other two salts my results are lower than those of Thovert, and I should be glad of the opinion of the Fellows as to the cause of this difference.

The research has been carried out in the laboratories of Birkbeck College, by the aid of a Government grant received through the Royal Society, and the author would like to record his indebtedness to Dr. Griffiths, head of the Physics Department at Birkbeck College, for his continued interest.

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ABSTRACT.

The Paper describes modifications made in the apparatus previously described (" Proc." Phys. Soc., XXIV., Dec. 15, 1911) to determine the value of the coefficient of diffusion of salts through water, by means of which the steady state is hastened and results obtained more quickly.

The single wide tube previously employed is replaced by a battery of shorter and narrower tubes. The error due to end-correction is investigated and results are given for the salts KCl, KNO_3 and NaCl for various concentrations down to very dilute solutions.

DISCUSSION

Dr. Griffiths said that he wished he could detect some flaw in Mr. Clack's methods, as he had always got higher results than those of the author. It appeared, however, to be much easier to get results too high than too low.

Mr. Paterson did not see why, in a symmetrical arrangement, such as the diffusion bulbs used by the author, the syphoning on which the maintenance of the concentration at the lower end of the diffusion tube depended should take place.

Mr. F. E. Smith wished to know if the author's method would be suitable for measuring the rate of diffusion of one solution into another of very nearly equal concentration and density. The problem was met with in standard cells. There was a slight difference in composition of the liquid throughout the cell, and this appeared to become practically constant after some years as though diffusion had ceased altogether.

Dr. Russell congratulated the author on having completed such a thorough investigation. When Glasgow University was removed to its new buildings at Gilmorehill in 1871, Lord Kelvin set up many long tubes containing liquids with the idea that the progress of the diffusion might be noted not only after years, but after hundreds of years. Although the author's earlier experiments had lasted months in some cases, he was now able to complete them in a few days.

The Author, in reply, thought it was an accident which way syphoning started, but when started it would obviously go on. It would be difficult to apply the method to solutions of nearly equal concentration, although it would measure the diffusivity of a strong solution with respect to a weaker one of the same salt to the same degree of accuracy as it measures that of a dilute solution with respect to water, the difference of concentration being the same in each case.

V.—*Note on the Conduction of Electricity at Point Contacts.*

By A. F. HALLIMOND, *Geological Survey and Museum, Jermyn Street.*

THE experiments here described were undertaken with a view to clearing up, if possible, one or two questions which suggested themselves during the examination of a number of minerals for "unilateral conductivity."

The subject has already given rise to a considerable literature, but as this has recently been very fully summarised in the "Proceedings" of this Society* references will here be given only to such Papers as bear immediately on the points under discussion.

The present Paper relates purely to the conductivity of "point contacts" when a steady, or slowly varying, E.M.F. is applied, and no attempt will be made to deal with the effect of high-frequency oscillations.

According to the usual practice the results are presented as "characteristic," or volt-ampere curves for the respective contacts. To obtain these curves it has been usual to take readings on ampere- and volt-meters, and from them to plot the characteristic on squared paper. Since, however, for the present purpose a large number of curves were required, a form of mirror galvanometer was used by which the characteristic was directly plotted as the path of the spot of light on the screen. The arrangement was of the ordinary rocking mirror type, and need not therefore be described in detail. One coil (of low resistance) was connected as amperemeter, the other as voltmeter. Thus the co-ordinates of the deflection were those of the characteristic curve.

An arrangement of this type is naturally not capable of the same accuracy as dial instruments, but, on the other hand, the curves obtained can be followed continuously, and any sudden change in the conductivity of the crystal contact is immediately indicated by a violent displacement of the spot of light. The time taken to plot one curve is about 15 seconds.

The first part of this note consists of a description of the behaviour of a typical point contact (zincite and tellurium) in terms of which the results for the other pairs can be described.

* P. R. Coursey, "Proc." Lond Phys. Soc., Feb., 1914.

The second half of the paper deals with the results of a systematic examination of the forty-five contacts which are possible between the ten substances employed. The question to be answered may be put briefly thus: If the behaviour of the contacts between substances A and B and A and C is known, can any rule be found for predicting the behaviour of the third combination B and C? A partial solution of the problem has been attempted by the arrangement of the ten substances in a series.

1. *The Behaviour of a Typical Point Contact (Zincite-Tellurium).*

The use of natural cleavage surfaces is not advisable, as these are frequently coated with a film of altered material. A polished surface of zincite was therefore employed, the fragment being imbedded in fusible metal, and the tellurium was in the form of a polished globule fused on the end of a pin. The pin was inserted in a lever which pressed it down on the zincite surface, different weights being hung on the end of the lever. For other substances which required lighter contacts the lever was replaced by a very light spring. The weights are given in grams at the contact.

The diagram (Fig. 1) shows the types of curve obtained. Under light weights the contact tellurium-zincite was relatively a non-conductor for voltages up to about 5 volts. As the weight increased (100 to 500 grams) the current passing with zincite negative became appreciable, increasing more rapidly than that with zincite positive. Thus the well-known unilateral curve was obtained. This is marked AA′ in the figure, and will be referred to as type A throughout this note.

With still further increase in weight the conductivity improved, but experiments in this direction were terminated by the breaking either of the tellurium globule or of the zincite.

Contacts such as galena-tellurium (Fig. 3), which give the A curve under a weight of 1 gram or less, become good conductors under increased weight (10 grams) and their characteristic then resembles that marked BB′; it is not linear, but the asymmetry is greatly reduced. It seems fair to assume that the same effect would be observed with zincite if the material were strong enough to stand the weight required.

An effort was now made to find the limits to which the A curve could be pursued on raising the voltage. It was found that when a certain point was reached (from 1 to 6 volts

according to the contact) the curve terminated in a sudden breakdown of resistance (AB, Fig. 1), and on lowering the voltage the tracing point returned along a much steeper curve, which passed through the origin at an angle corresponding with a resistance of 500 ohms or less, and could be pursued to some distance beyond it (BB′). The voltage at which this breakdown occurred was on an average higher with zincite positive than when it was negative. This curve is marked BB, and will be termed type B. It persists, at any rate for some hours, in most contacts, and it seems clearly due to a definite change of condition due to the breakdown.

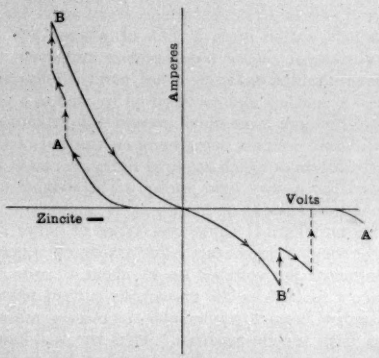


FIG. 1.—THE DIAGRAM REPRESENTS THE TWO TYPES OF CURVE COMMONLY ENCOUNTERED.

The curve marked A A′ is the common unilateral curve, B B′ is the result of applying excessive voltage, and these are respectively termed types A and B throughout the Paper.

The dotted lines indicate the movement of the spot of light on breakdown and on recovery. The breakdown is sudden, the recovery is relatively slow.

This curve BB in its turn was now examined under increasing voltage. With zincite negative it underwent further alterations, becoming more and more linear in form, with decreasing resistance. If it had been caused by a breakdown of the A curve with zincite positive it generally remained permanent, but if the original breakdown had occurred with zincite negative, a very remarkable recovery could be obtained in the following manner.

If the voltage was raised to a certain point (generally from 1 to 4 volts) with zincite positive and the contact was then allowed to stand, the current could very frequently be seen to diminish more or less rapidly, so that the spot of light returned toward the A curve (point B′, Fig. 1). In favourable circumstances the A curve was actually regained and could then be repeated in its original form. If the recovery was partial the resulting curve was of an intermediate character throughout its course, as indicated by the short curves in the figure.

The following table of readings on an amperemeter will give some idea of the progress of such recoveries.

Contact Copper-zincite, standing after alteration with Zincite Positive. (*E.M.F., approx.* 2.5 *volts.*)

Time.	Current.		
	Expt. *a.*	*b.*	*c.*
0	...	...	...
10	215	186	210
20	200	180	202
30	198	179	197
40	194	178	189
50	191	177	184
60	189	173	182
70	184	170	...
80	180	168	...
90	184	168	172
100	182	170	172
Seconds.		Micro-amperes.	

2. *Comparison of the Results given by Polished Contacts of Ten Substances taken in Pairs.*

Curves of the type described above were obtained in every case,* though not with the same facility, for in some cases the contacts required were extremely light.

The great similarity of the unilateral curves is well known to all who have had occasion to plot curves for various contacts, it is illustrated in the two curves reproduced in Figs. 2 and 3. Since the curves, especially the A type, are strongly unilateral in character, it is possible, for the purposes of classification, to describe the effect by giving the signs + and — respectively

* It will be observed that all these curves are convex toward the axis along which the voltage is measured. The theory recently discussed by Eccles ("Proc." Lond. Phys. Soc., June, 1913) requires for many cases curves which are concave. No such curves were observed in any of the contacts here described.

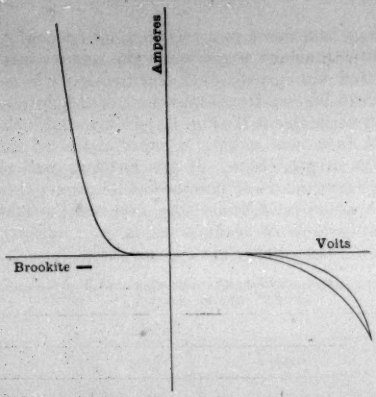


FIG. 2.—CURVE OF TYPE A FOR THE CONTACT OF BROOKITE WITH GALENA.

The B type does not seem to be obtainable by pressure, as neither brookite nor zincite will withstand the increase of weight which might be expected to yield it. The weight for the curve shown is already 350 grams.

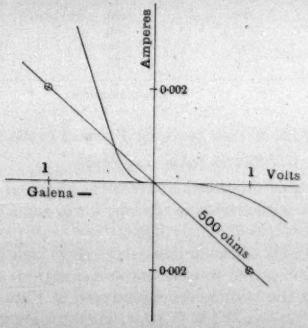


FIG. 3.—CURVE OF TYPE A FOR THE CONTACT OF GALENA WITH TELLURIUM.

The similarity of this with Fig. 2 can leave little doubt that the same effect is being observed, although the weight is here of the order of 1 gram. The B type is readily obtained on increasing the weight, or by application of excessive voltage.

The straight line illustrates a direct method of ascertaining the scale; it is the "characteristic" of a 500-ohm voltmeter. The scale of the other figures is approximately that shown here.

None of these curves show irregularities greater than would be expected from the errors of tracing, except where definite breakdown has occurred.

to the two substances in contact; thus, quite arbitrarily, zincite might be called negative to tellurium, tellurium positive to zincite.

When this has been done a table can be prepared, such as that below, in which the sign given in each square describes the curves given by the corresponding mineral in the top line when in contact with the mineral named in the column on the left.

In this table, as first drawn, the distribution of the negative signs was at first sight quite arbitrary, but by changing the order of the minerals named it was found that all the negative signs could be brought below the diagonal as in the following table :—

——	Zincite.	Brookite.	Molybdenite.	Chromium.	Galena.	Iserite.	Chalcocite.	Copper.	Chalcopyrite.	Tellurium.
Zincite										
Brookite	—									
Molybdenite	—	—								
Chromium	—	*	—							
Galena	—	—	—	—						
Iserite	—	—	—	—	—					
Chalcocite	—	—	—	—	—	—				
Copper	—	—	—	—	—	—	—			
Chalcopyrite	—	—	—	—	—	—	—	—		
Tellurium	—	—	—	—	—	—	—	—	—	

From this it is at once evident that the 45 results can be expressed by means of a series of the ten substances, such that the higher member behaves toward the lower as zincite to tellurium.

It should be said that the substances used were selected as typical and uniform in composition. It is most probable that another set of even the same minerals, taken from other localities, would yield a somewhat different series. Further,

* Decisive results were not obtained for this contact.

the effect may be dependent on the orientation of the planes in contact. For these reasons no attempt has been made to give either the locality or the composition of the minerals used; they merely represent certain samples of material.

3. *Conclusions.*

Comparison of these results throws some light on the nature of the effect represented in the unilateral curves.

In the first place (to choose an example which is for the most part established by results published by previous observers) the characteristics given by all the substances here examined, when in contact with zincite under a weight of 1,000 grams, were almost identical in shape. It therefore seems fair to assume that the unilateral effect here obtained is a property of the zincite, and is independent of the physical constants of the other substance used.

The same reasoning may be applied to the contacts of, say, galena with substances standing below it in the table.

Now it is evident that the same substance (*e.g.*, galena) may play one of two distinct parts in a point contact. Toward substances higher in the series, such as zincite, it is in the position which has been called positive. Toward substances below it, such as tellurium, it is negative. What, then, is the condition which determines its behaviour? The answer to this question seems to lie in the fact that, as far as the very rough measurements go, the weight† required to obtain the unilateral curve diminishes progressively along the series, from about 1,000 grams for the zincite contacts to extremely light pressure for the contact of chalcopyrite with tellurium.

This being the case, it is evident that the substances below zincite, when under weights of 1,000 grams, will act alike as good conductors, the resistance of the contacts with zincite being only that of the zincite surface layer in every case.

† It will be noted that weights, not pressures, are here given. In these contacts the softer substance is spread over a certain area on the harder surface. Thus for a given weight the area increases as the substance is softer, but the pressure is correspondingly diminished. If the conductivity of the contact depends jointly on the area and pressure, it will then be roughly independent of the hardness of the substances in contact. The fact that the curves were alike for contacts of zincite with chromium and zincite with molybdenite under 1,000 grams, the area of the latter being about 1 square millimetre, renders it clear that some such assumption is necessary, though the statement that the conductivity is directly proportional to the pressure can only be taken as a rough approximation to the true relationship.

Under lighter weight galena, say, in its turn possesses a high surface resistance which yields a unilateral curve in contact with tellurium.

Many experiments will suggest themselves by which these conclusions could be tested, but it is impossible to discuss them within the limits of this note, of which the chief aim is to present a plea for the examination of the effect on strictly experimental lines. The serial arrangement suggested is advanced as a tentative solution to the question put forward on p. 76. Whether it survives further examination or not it may perhaps serve as a basis for the discussion of the large number of "rectifying" contacts now known.

ABSTRACT.

The Paper deals with the "characteristic" or volt-ampere curves given by various "Point" contacts when the voltage is slowly varied. The curves were plotted by means of a form of rocking mirror galvanometer, which projected the characteristic as the path of a spot of light on the screen, the co-ordinates being respectively proportional to the current and voltage.

The first part describes the behaviour of a typical contact, zincite-tellurium. The well-known unilateral curve is terminated by a sudden breakdown of resistance, after which the contact has a more symmetrical characteristic of lower resistance at the origin. By allowing the contact to stand under a certain voltage with zincite positive it may frequently be restored to the condition of high resistance in which it again yields the unilateral curve.

The second part describes the results obtained on examining the characteristics for the forty-five contacts possible between 10 chosen substances. The conclusion is reached that the results in all cases are similar to those given by zincite-tellurium. No line could be drawn separating "metallic" contacts from those in which one or both conductors were "crystals."

The results obtained are expressed in terms of a series in which the higher member behaves towards the lower as zincite to tellurium.

Series: Zincite, brookite, molybdenite, chromium, galena, inserite, chalcocite, copper, chalcopyrite, tellurium.

In the third part the conclusion is drawn that in a contact yielding the unilateral (high resistance) curve, the resistance lies within the surface of the member standing higher in the series. Rough measurements showed that the weight required in these contacts diminished from about 1,000 grams for zincite to very light contact for the substances lying near tellurium, and it is suggested that this gradation in weight determines the relative positions arrived at for the respective substances.

DISCUSSION.

Mr. DUDDELL thought the author's method of obtaining continuous curves much more satisfactory than the ordinary method, as it gave less chance of the point contacts altering during the experiment, and, further,

the arrangement of the minerals in the table he gave appeared to have some correspondence with their order of sensibility when used as wireless detectors.

Mr. D. Owen attributed much of the success of the author's experiments to the attention paid to the preparation of polished surfaces of contact. It would be useful if the author would specify the curvatures employed. The two types of characteristic might be accounted for by the greater or less extent to which the electrostatic attraction between the contacts extend, type B occurring when that effect was large. Calculation proved that, assuming a P.D. of only one volt, and a distance apart of 10^{-7} cm. (ten times the molecular distance), the electrostatic pull was of the order of 1,000 gm. *wt./mm*^{2}. It might in some cases be much greater than this.

Mr. P. R. Coursey thought the main point of interest in the Paper was the table in which the substances were arranged in the order of the effects obtained with them. It would be of considerable use if some connection could be found between the arrangement of the substances in this table and some of their chemical or physical properties, as it should then be possible to extend the present limits of the table to include new substances, and in this way, it may be, to obtain perhaps even more sensitive wireless detectors than were at present available.

The visual method of taking the characteristic was of value for rapid investigation of the behaviour of any particular contact, and in this connection he would ask the author whether in his experiments employing this method he obtained any evidence of a "hysteresis" effect in the contact. A number of experimenters had stated that such an effect does exist, and it should be an easy matter with this arrangement definitely to decide the point for any contact or specimen.

With reference to the jumping of the characteristic from the "A" to the "B" type of curve by the application of excessive voltage, it seemed evident that this must be what happened when a strong "X" was received by a detector; but it does not, however, seem to explain an effect that he had noticed (and others had obtained similar results) when using a zincite-tellurium contact as a detector, that the impact of a strong atmospheric very often does not spoil the reception of signals, but sometimes they are actually stronger after it has passed. This may occur several times in succession, each time the strength of the signals rising somewhat, till finally a steady state seems to set in, in which signals are much louder than it was possible to obtain at first.

The author mentions in the Paper that on no occasion has he obtained characteristic curves *concave* to the volt axis, although Dr. Eccles's theory indicates that such should be possible. On one occasion he had obtained a curve of a particular zincite-bornite contact that was concave to the volt axis on the side with zincite positive, but was convex with the zincite negative, while, moreover, it seemed to be a stable condition as the curve could be repeated (slide shown). It appears, however, to be rather a peculiarity of the specimens employed than a property of the materials, as with other samples he had failed to obtain any such result. It seemed probable, too, that if the voltage limits had been taken higher the curve might have turned and become convex to the volt axis for the higher values.

Prof. Fortescue communicated the following: The curves given in Fig. 1 are very interesting. I have repeatedly noticed the same thing with the zincite and galena series of contacts. The breakdown is apparently due to increase of area of the contact, brought about by the increase of mechanical pressure or by the softening, or even fusing, of one of the materials when high voltages are applied. The restoring effect of the reverse current is not so easy to explain; possibly it may be some electrolytic action. I have never found a point which had been rendered conductive by heavy mechanical pressure restored in this way. The

tabulation of the substances given on page 2 can also be carried out from their thermo-electric properties with similar results. Carborundum, however, presents a difficulty, as it can be strongly positive or negative, and is to a large extent independent of the mechanical pressure.

The AUTHOR, in reply, said that it seemed likely that some correspondence would exist between the order of the substances in the table and the behaviour of the contacts as detectors. If the conclusion of Pierce is accepted, that the characteristic is still followed under high frequencies, the point of best sensitiveness for weak signals should lie near the maximum curvature. The discussion of this question is scarcely possible within the limits of the present note. The ease of adjustment is an important factor in practice; it seems dependent on two conditions: (1) The breakdown may supervene before the bend in the A curve is reached. This is especially true of the metals. (2) The weight permissible for a given contact pair lies between a lower value, for which both surfaces are poor conductors, and an upper limit at which they conduct yielding a relatively symmetrical curve. The further apart the bodies are in the series the greater is the range of weight over which they give the A curve, and the greater, therefore, is the ease of adjustment.

He had not so far been able to connect this series with one derived from other physical properties. Hysteresis, such as was described by Pierce, was observed. This, however, he would assign to alteration and recovery. If the breakdown in Fig. 1 is made less marked, the curve becomes extremely similar to the hysteresis curve. In some experiments the change was so slight that even with the mirror the step was not obviously discontinuous. The action of an X on the detector can scarcely be predicted from the change in the characteristic alone. It would be interesting to see the nature of the characteristics immediately before and after such an alteration. If the X was strongly damped the first impulse might tend either to breakdown or to recovery. The curve shown by Mr. Coursey was extremely interesting. He had traced several curves for zincite-bornite, but had not encountered this form.

The curvature of the surfaces appeared to be immaterial. The form of the area of contact would apparently be that of a small element of the harder surface. Provided the radius of curvature were not comparable with the diameter of contact, the area would be virtually plane; for this reason no special shape beyond that of a blunt point was attempted. Different considerations might, of course, apply to the use of splintered points of mineral. The electrostatic attraction does not seem to offer of itself any possibility of a unilateral effect, since the force on interchanging + and − signs would be the same. Some unilateral property would still be required. In the case of zincite the area in most cases was nearer 0·01 sq. mm. than 1·0. This would require a higher potential gradient, of the order 10^9 volts per centimetre. As the dielectric strength of mica is only 2×10^8 volts per centimetre, the strain in the intervening space would be considerable. If the electrostatic pull were the direct cause of conductivity in the B curve, the curves A and B should be tangential at the origin, for here the pull has a negligible effect on the natural resistance of the contact. As far as could be seen, the two curves cut at a finite angle. Possibly the mechanical breakdown of some film might be assumed, but the high weights in zincite contacts offered difficulties.

The recovery of the contact was undoubtedly one of the most difficult features of these curves. It seemed impossible to conceive a mechanical change which would be reversible to such an extent. The writer felt that the published thermo-electric data which he had encountered were too few and conflicting to permit a comparison with any degree of certainty. The behaviour of such a mineral as pyrite, which exists in two widely different thermo-electric forms, should yield interesting information.

The very high weights required by carborundum would suggest that it

should behave in the same way as zincite, possibly lying beyond it in the table. A very good unilateral curve was published by Pierce,* and this curve has the high current with carborundum negative, "current (in the crystal) toward the platinised surface," as would be expected. The difficulty with carborundum may arise in some cases from the incomplete contact with the fusible metal, and also from the possibility that many of the curves under light pressures were very probably of the B type. Like cassiterite,† carborundum‡ appeared to be a colourless substance, owing its colour and possibly its conductivity to dissolved matter. The variations in the electrical properties of such bodies were probably very great.

* G. W. Pierce, THE ELECTRICIAN, December, 1907, p. 375.
† Th. Liebisch, "Sitzungsber. d. K. Preuss. Akad. d. Wiss," 1911, 414.
‡ "Zeit fur Kryst," L., 33.

VI. *Thermal Conductivity. Part II.: Thermal Conductivity of Badly-conducting Solids.* By THOMAS BARRATT, *A.R.C.S., B.Sc.*

RECEIVED OCTOBER 13, 1914.

I. Introduction and Historical.

A NEW method of measuring the thermal conductivity of solids was recently described by the author,* and the method was applied to the determination of the conductivity of certain of the rarer metals and alloys at air temperatures and at 100°C.

The same arrangement—with one or two modifications—has now been employed in the measurement of this property in the case of badly-conducting solids. The results are given in detail in the following Paper. A number of typical substances have been tested—*e.g.*, woods, both hard and soft; common electrical insulators, such as glass, fused silica, ebonite, and firebrick; and in addition, carbon, which is a partial conductor of electricity. The results indicate that the method adopted is as successful in the case of these substances as it proved to be when pure metals and alloys were under investigation. Perhaps the most interesting point in connection with these measurements is the fact that precisely the same method, and in the main the same apparatus, have been employed both for metals and non-metals. Up to the present time the methods adopted for the determination of the thermal conductivity of non-metallic substances have been essentially different from those used in the case of metals, so that the present series of experiments affords an opportunity for the first time of a direct comparison of the conductivities of the two classes of solids.

The experiments of Lees,† of Jäger and Diesselhorst,‡ and of the present author§ have shown that the thermal con-

* T. Barratt, "Proc." Phys. Soc., XXVI., Part V., pp. 346-371, Aug., 1914.

† C. H. Lees, Bakerian Lecture, Roy. Soc., 1908.

‡ W. Jäger and H. Diesselhorst, "Abh. d. Phys. Tech. Reich.," 3, 269, 1900.

§ T. Barratt, *loc. cit.*

ductivities of some of the pure metals—notably bismuth and lead—are of the same order of magnitude as those of some of the non-metals. For example, quartz and graphite gave results considerably higher than those of bismuth (Lees, and Barratt, *loc. cit.*). In the following investigation the results obtained for bismuth and lead by Jäger and Diesselhorst have been adopted as a starting point, and the conductivities of the non-metals deduced by direct comparison.

Of the methods employed up to the present time, the disc arrangement of Lees and Chorlton,* and of Lees,† is perhaps the most accurate and best known. This method has been used, with unimportant modifications, by several experimenters, including A. Eucken,‡ some of whose results are alluded to in a later section of the present Paper. (*See* Section 5.)

Another arrangement, similar in some respects to that of Lees, was adopted by Biquard, for the measurement of the thermal conductivity of heat insulators. The material was in the form of a circular plate, surrounded by a guard-ring of copper, thus securing a normal flow of heat, this being supplied by a current of warm water maintained at a constant temperature. On the plate and guard-ring were placed concentric boxes full of broken ice. The rate of melting of the ice gave an indication of the heat supplied. The method is in some respects similar to that employed by Berget for the conductivity of metals, and shares with it the disadvantage that the temperature of the side of the plate next the warm water is not the same as that of the water. Again, the presence of the metallic thermometers employed interferes considerably with the uniformity of the flow of heat through the specimen, and with its temperature as given by these thermometers. In the method described in the present Paper the specimen is completely bare from end to end, the platinum thermometers employed being in positions quite removed from the solid itself.

A totally different arrangement was that employed by Clement and Egy § in the measurement of the thermal conductivity of fireclay. The substance was in the form of a

* Lees and Chorlton, "Phil. Mag.," [5], 41, 495, 1896.
† C. H. Lees, "Phil. Trans.," 191, 399, 1898.
‡ A. Eucken, "Ann. d. Physik.," 34, pp. 185-221, February 3, 1911.
§ J. K. Clement and W. L. Egy, "Met. and Chem. Eng.," 8, pp. 414-416, July, 1910.

cylinder, 40 cm. long, 12 cm. diameter, and was heated electrically by a nickel wire spiral passing through a hole along its axis. The temperatures at various distances from the axis were measured by thermo-couples, whose presence must lead to the same distortion of the flow of heat as noted above in Biquart's method.

A somewhat similar method, again, was employed by R. Melmer,* in the case of substances such as fats, earth, and sands. A thin central platinum tube, within which was a mercury thermometer, was heated by an electric current. Concentric with this was a wider tube of brass, the substance being tightly packed in between the two tubes. Measurements were made of the heat supplied and of the temperatures of the platinum and the brass, and the thermal conductivity deduced.

2. *Arrangement of Apparatus.*

The substances tested were in the form of accurately turned cylinders of diameter 5 mm. or 6 mm., and of lengths from 5 cm. to 20 cm. Each cylinder was lightly and uniformly painted over with a dead-black varnish, consisting of an aniline dye dissolved in amyl acetate. One end of the cylinder AC (Fig. 1)

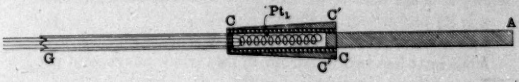


FIG. 1.

fitted accurately into a slightly conical hole bored in a copper cylinder CC, of length 4 cm., inner diameter 6·5 mm. Around CC was wound 3 metres of single silk-covered pure platinum wire, gauge 30. Around this coil of wire a single layer of shellac varnished silk paper was pasted. Over this again was another copper cylinder, C'C', whose thickness increased from zero at one end to about 2 mm. at the end into which the cylinder AC fitted.

Inside the inner cylinder was a platinum thermometer Pt_1, which gave the temperature of the end C of the cylinder AC. A thin glass tube CG served to keep the apparatus steady and in the centre of a water or steam jacket which enclosed the

* R. Melmer, "Akad. Wiss. Wien.," Sitz. Ber. 120, pp. 269-281, March, 1911.

whole apparatus, and maintained its temperature constant. This temperature was registered by a second platinum thermometer, Pt_2, which rested within the outer jacket just opposite the middle point of the specimen AC. The difference of temperature (V) between the "hot" end of the cylinder AC and the enclosure was thus given by the difference of the readings (Pt_1—Pt_2) of the two platinum thermometers. The ends of the platinum wire wound round the inner cylinder were connected to a carefully calibrated potentiometer, which, in connection with a standard one-ohm coil, enabled one to obtain the current (C) through the wire, and the E.M.F. (E) at its ends. The arrangement of the potentiometer, &c., was precisely the same as that described in the Paper already referred to on the "Thermal and Electrical Conductivities of some of the Rarer Metals and Alloys." (T. Barratt, *loc. cit.*)

It was shown by the author in this Paper that the thermal conductivity "k" of a substance under the above conditions is given by

$$k=\frac{H^2 \coth^2 al}{pqhV^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where H is the heat received by the cylinder at its hot end,

p is its perimeter,

q its cross-sectional area,

h the heat lost from 1 sq. cm. of its surface per second per 1°C. difference from its surroundings,

V the difference of temperature between the hot end of the cylinder and the enclosure,

l the length, and

$$a=\sqrt{\frac{hp}{qk}}.$$

When l is long enough, this expression reduces to the very simple form

$$k=\frac{H^2}{pqhV^2}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

and the latter form of the equation has been employed in nearly every case treated in the present series of experiments.

A simple proof of this latter equation, from first principles, may be of interest.

Let the cylinde be placed in a gas at zero temperature.

The amount of heat per second through a plane at distance x from the hot end (which is at temperature V) is

$$-kq\frac{dv}{dx}.$$

The heat through a parallel plane at distance $x+dx$ is

$$-kq\frac{dv}{dx}+\frac{d}{dx}\left(-kq\frac{dv}{dx}\right)dx.$$

Hence the heat lost from the surface (of area $p\,.\,dx$) between these planes is

$$kq\frac{d^2v}{dx^2}dx=pdxhv.$$

Multiply each side of this equation by $2\frac{dv}{dx}$ and integrate.

$$\therefore \qquad kq\left(\frac{dv}{dx}\right)^2=hpv^2+\text{A}.$$

If the length of the cylinder is such that a part of it is at the temperature of the enclosure, we have, at points where $v=0$,

$$\frac{dv}{dx}=0.$$

Hence $$\text{A}=0,$$

and $$kq\left(\frac{dv}{dx}\right)^2=hpv^2. \qquad \text{(A)}$$

This equation is applicable at every point of the cylinder. At the hot end, where $v=\text{V}$,

$$\text{H}=-kq\frac{dv}{dx}. \qquad \text{(B)}$$

Hence, from (A) and (B),

$$qk\frac{\text{H}^2}{q^2k^2}=hp\text{V}^2,$$

or $$k=\frac{\text{H}^2}{pqh\text{V}^2}.$$

The amount of heat (H) given to the specimen cylinder is equal to $\frac{\text{CE}-\text{C}'\text{E}'}{\text{J}}$, where C, E, are the current and E.M.F. respectively with the specimen in place; and C′E′ are corresponding quantities when the specimen is replaced by a copper

plug whose end is flush with the end of the copper cylinder, CC, the excess of temperature, V, being maintained the same in both cases. A correction, which is in some cases a comparatively large one, has to be made for the heat lost from the exposed end of the copper plug. This is done by adding to " H " as calculated above a quantity (H′) obtained by the product of q, h and V. The accuracy attainable in this correction is, however, just as great as that obtained in the main experiment.

In performing an experiment, the specimen is put into position, the current turned on, and the apparatus left for one or two hours till conditions have become steady, when C, E, and V are carefully measured. The cylinder is then removed, the copper plug substituted, and the current adjusted so as to keep V the same as before. (This latter operation takes only a few minutes, so that little or no alteration of temperature takes place within the enclosure.) C′ and E′ are then quickly determined. The arrangement of the electrical part of the apparatus is the same in all respects as that employed by the author in the case of metals, in the Paper referred to above, and it is therefore unnecessary to describe it in detail.

3. *Determination of* " h."

In order to determine the amount of heat, " h," lost per second by 1 sq. cm. of surface of the substance when its temperature is 1°C. above that of its surroundings, it is clear that in consequence of the very low electrical conductivity of the specimens employed, a direct method is inconvenient, if not impossible.

From equation (1), however, we obtain

$$h=\frac{\mathrm{H}^2 \coth^2 al}{pqk\mathrm{V}^2}.$$

Assuming Jäger and Diesselhorst's results for the thermal conductivity of bismuth or lead, the remaining quantities on the right-hand side of this equation can be obtained by the same series of measurements as are made in the main experiments. This has therefore been done in the case of pure lead and of pure bismuth at temperatures 20°C. and 100°C., and the mean results of a great number of experiments gave the values of " h " as follows :—

" h " at 20°C. = 0·000224 ; " h " at 100°C. = 0·000265.

Below are recorded the measurements made in two of these experiments.

BISMUTH.
July 7th, 1914. Temperature 20°C.
Radius, $r=0{\cdot}300$ cm., $pq=2\pi^2r^3=0{\cdot}533$; $k=0{\cdot}0194$.
$V=Pt_1-Pt_2=9{\cdot}190$°C., $H'=qhV=0{\cdot}00058$.

1. With bismuth rod in place.	2. With copper plug in place.
$Pt_1-Pt_2=9{\cdot}190$°C.	Pt_1-Pt maintained 9·190°C.
*E=6283	E′=5122
*C=388·7	C′=317·2
†W=3898	W=3898

* C and E are readings on potentiometer, in ohms, corresponding to current through wire, and E.M.F. at its ends, respectively.
† W is the reading when a Weston cell is in place.

The true "E" is obtained from the expression E/W × E.M.F. of Weston cell, and similarly for true current "C."

Hence $H=(CE-C'E')/J=0{\cdot}01334.$

$H'=qhV=0{\cdot}00058.$

True "H"$=0{\cdot}01392$, and $H^2=0{\cdot}0001938.$

"h" at 20°C.$=(H^2\coth^2 l)/(pqkV^2)=0{\cdot}000223$

LEAD.
July 11th, 1914. Temperature 100°C.
Radius $r=0{\cdot}267$ cm., $pq=0{\cdot}3756$; $k=0{\cdot}082$.
$V=7{\cdot}044$°C.; $H'=qhV=0{\cdot}00047$.

1. With lead in place.	2. With copper plug in place.
$Pt_1-Pt_2=7{\cdot}044$°C.	Pt_1-Pt_2 maintained 7·044°C.
E=7153	E′=5263
C=365·8	C′=269·2
W=3897	W=3897

Hence $H=0{\cdot}01963$; $H'=0{\cdot}00047$.

True $H=0{\cdot}02010$; $H^2=0{\cdot}0004040.$

and "h" at 100°C.$=(H^2{}_0\coth^2 al)/(pqhV^2)=0{\cdot}000265.$

The values obtained for "h" at 100°C. in the case of bismuth showed good agreement with those for lead; and, similarly, for both metals at 20°C. This affords a verification of the values for the thermal conductivity of these metals as given by Jäger and Diesselhorst in the Paper already referred to. The rods of bismuth and lead employed in the experiments detailed above were of sufficient length to make the value of $\coth^2 al$ almost negligible in each case.

4. *Determination of Thermal Conductivity, "k."*

In order to determine the thermal conductivity, k, the procedure already detailed in sections 2 and 3 was followed. In the calculation of "k" from these measurements the values of "h" previously obtained, viz. 0·000224 at 20°C. and 0·000265 at 100°C., were substituted in the equation, $k=(H^2 \coth^2 al)/(pqhV^2)$, and the thermal conductivity, k, at once deduced.

Below are given full details of two of the experiments, together with the method of reducing the results.

Red Fibre. July 15th, 1914. Temperature 20°C. Radius $r=0{\cdot}336$ cm. ; $pqh=2\pi^2r^3h=0{\cdot}0001677$	
1. Specimen in place. $Pt_1-Pt_3=V=10{\cdot}191$°C.	2. Specimen replaced by copper plug. V maintained at 10·191°C.
E=5687	E′=5360
C=3495	C′=329·5
W=3900	W=3900
E.M.F. of Cadmium Cell=1·0186 volt.	

$$\text{True current}=C/W\times 1{\cdot}0186.$$

$$\text{True E.M.F.}=E/W\times 1{\cdot}0186.$$

$$\text{Heat given (H)}=(EC-E'C')/JW^2\times(1{\cdot}0186)^2+qh\text{V}.$$

$$=0{\cdot}003609 \text{ calories.}$$

Hence "k" at 20°C.$=(H^2 \coth^2 al)/(pqhV^2)=0{\cdot}00112$.

Red Fibre. Sept. 8th, 1914. Temperature 100°C. Radius $r=0{\cdot}336$ cm., $pqh=0{\cdot}0001984$.	
1. Specimen in place. $Pt-Pt_1=V=10{\cdot}57$°C.	2. Specimen replaced by copper plug. V maintained at 10·57°C.
E=7060	E′=6693
C=359·7	C′=340·1
W=3879	W=3879

Heat given $=(EC-E'C')/(JW^2)\times(1{\cdot}0186)^2$; $qh\text{V}=0{\cdot}005143$ calories.

k at 100°C.$=(H^2 \coth^2 al)/(pqhV^2)=0{\cdot}00119$.

In the accompanying table (Table I.) are given the experimental details relating to the measurement of the thermal conductivity of all the metals tested, both at air temperatures and at 100°C.

TABLE I.—*Thermal Conductivity of Non-metals at* 20°C. *and* 100°C.

Substance.	Date.	Temp.°C.	Radius (r) (cms.)	qh $\times 10^7$	pqh $\times 10^7$	E (ohms)	C (ohms)	E′ (ohms)	C′ (ohms)	W (ohms)	H(cals) $\times 10^6$	V (Cent.)	Length (cms.)	Density gms.p.cc	$k \times 10^4$
Fused Silica	July 27	20°	0·332	775	1,618	5,886	369·2	5,433	340·4	3,890	6,037	9·75	6·1	2·17	23·7
	Aug. 26	100°		917	1,914	7,140	363·2	6,584	334·7	3,879	7,385	10·57			25·5
Soda Glass	July 27	20°	0·332	775	1,618	5,699	357·2	5,340	331·5	3,890	5,095	9·64	6·0	2·59	17·2
	Aug. 26	100°		917	1,914	7,044	358·3	6,584	334·7	3,879	6,242	10·57			18·2
Red Fibre	July 15	20°	0·336	794	1,677	5,687	349·5	5,360	329·5	3,900	4,420	10·19	5·9	1·29	11·2
	Sept. 8	100°		1,039	1,984	7,060	358·7	6,693	340·1	3,876	5,413	10·57			11·9
Fire-Brick	Sept. 12	20°	0·336	794	1,677	5,924	369·5	5,614	350·0	3,874	4,541	10·57	5·5	1·73	11·0
	Sept. 2	100°		1,039	1,984	6,985	354·8	6,651	337·9	3,879	4,900	10·57			10·9
Ebonite ...	Sept. 12	20°	0·327	752	1,546	5,703	355·1	5,635	351·4	3,874	1,535	10·57	6·3	1·19	1·36
	Sept. 2	100°		889	1,829	6,728	341·9	6,653	338·0	3,879	1,789	10·57			1·31
Gas Carbon	July 25	20°	0·251	443	699	5,932	370·2	5,337	332·9	3,892	7,279	9·51	14·1	1·42	85·0
	Aug. 27	100°		524	827	7,329	372·5	6,574	334·0	3,880	9,354	10·57			95·0
Mahogany	Sept. 14	20°	0·325	743	1,517	5,633	350·6	5,459	339·5	3,874	2,936	10·57	5·5	0·55	5·09
	Aug. 25	100°		879	1,795	6,876	349·2	6,630	336·9	3,880	3,760	11·42			6·05
Satin Walnut (along grain)	July 28	20°	0·325	743	1,517	5,692	352·9	5,536	342·4	3,891	2,603	10·10	5·9	0·50	4·36
	Aug. 29	100°		879	1,795	6,798	345·1	6,586	334·4	3,879	3,296	10·57			5·42
Satin Walnut (across grain)	July 29	20°	0·332	775	1·618	5,474	341·4	5,398	336·6	3,892	1,633	10·13	4·8	0·50	1·60
	Aug. 29	100°		917	1,914	6,706	340·5	6,586	334·4	3,879	2,304	10·57			2·48
Oak	July 27	20°	0·325	743	1,517	5,521	345·5	5,336	332·8	3,892	3,029	9·61	6·3	0·65	5·83
	Aug. 31	100°		879	1,795	6,843	347·5	6,617	335·9	3,857	3,485	10·57			6·07
Lignum vitæ.....	July 21	20°	0·334	785	1,647	5,693	349·6	5,476	336·5	3,898	3,206	10·16	6·8	1·16	6·04
	Sept. 1	100°		928	1,949	6,881	349·2	6,617	335·9	3,879	3,948	10·57			7·16
Greenheart	July 21	20°	0·334	785	1,647	5,847	364·4	5,534	344·6	3,894	4,468	10·39	7·5	1·08	11·2
	Sept. 1	100°		928	1,949	6,958	353·5	6,617	335·9	3,879	4,888	10·57			11·0
American whitewood	July 13	20°	0·325	743	1,517	6,384	387·1	6,210	377·2	3,901	3,105	12·49	6·5	0·575	4·07
	Sept. 3	100°		879	1,795	6,838	346·7	6,648	337·7	3,880	2,998	10·57			4·48
Boxwood	July 28	20°	0·327	752	1,546	5,742	356·0	5,592	346·6	3,891	2,408	10·28	5·7	0·901	3·56
	Sept. 10	100°		889	1,829	6,858	348·5	6,684	339·7	3,879	2,927	10·57			4·14

The results have all been corrected in the following particulars :—

1. Calibration of coils of resistance boxes used in potentiometer.

2. Standardisation of the one-ohm resistance coil and of the three cadmium cells used.

3. Calibration of coils of platinum resistance bridge, and reduction of platinum temperatures to the hydrogen scale.

The instruments used were precisely the same as those employed in the determination of the thermal conductivity of pure metals and alloys in the Paper mentioned before.

The correction due to the " resistance at the joint " between the specimen and the copper block into which it fitted is eliminated in the present series of experiments, being allowed for in the determination of " h," when a bismuth or lead cylinder is employed. It may be assumed that the " joint resistance " is the same in the case of the bismuth or lead as when the other (similar) cylinders are substituted in their place under precisely the same conditions.

5. *Discussion of Results.*

The values obtained in the present Paper for the thermal conductivities of non-metals agree in most cases very well with those recently obtained—that is, where corresponding results are available. In Table II. a comparison is made of these quantities.

TABLE II.

Substance.	Thermal conductivity. (Present Paper).	Thermal conductivity. (Previous results).
Soda glass.....	0·00172 at 20°C. 0·00182 at 100°C.	0·0013 to 0·0018 (K and L) 0·00227 (12° to 32°) (P).
Ebonite.....	0·000136 at 20°C 0·000131 at 100°C.	0·00042 (L). 0·00038 at 90°C. 0·000089 (F).
Gas Carbon.....	0·0085 at 20°C. 0·0095 at 100°C.	0·0012 (K and L).
Mahogany.....	0·000509 at 20°C.	0·0005 (L).
Oak	0·000605 at 100°C. 0·000583 at 20°C. 0·000607 at 100°C.	0·0006 (K and L).
Firebrick.....	0·00110 at 20°C. 0·00109 at 100°C.	0·0024 (H, L and D)

K and L refers to Kaye and Laby's Tables, pub. Longmans, Green & Co., 1911.

L: C. H. Lees, *loc. cit.*

P: O. Paalhorn, Diss. Jena, 1894.

D: A. Dina, Rend. Inst. Lombardo, 32,205, 1899.

F: Forbes, " Proc." Edinb. Soc., 8, 62, 1874-75.

H L and D: Herschel. Lebour and Dunn, Rep. Brit. Soc., 49, 1879.

It is to be expected that values obtained for similar substances should differ rather widely, as it is obviously impossible to obtain specimens identically alike of any particular substance of the classes dealt with here. Even in the investigation of thermal conductivities of the metals most observers have noted the marked influence of even a minute percentage of impurity. This influence is, of course, much more powerful in the cases under investigation in the present research.

It appears from the results that most non-metallic substances conduct heat more readily at higher temperatures. The same result has been noted by several other observers, notably Clement and Egy,* in the case of various samples of fireclay; and Eucken† in investigations on the conductivity of amorphous bodies. The latter observed (as in the experiments of the present author) that the thermal conductivity of ebonite did not change much with the temperature. The same observation may be made of the conductivity of "red fibre" and of firebrick.

The results of the present series of experiments seem to indicate that the conductivity of most kinds of wood increases very rapidly indeed with the temperature. Too much significance, however, must not be attached to this, as continual heating at 100°C. probably has the effect of altering very considerably the structure of the wood. Experiment showed that constant results at 100°C. could not be obtained until the wood had been kept at this temperature for several hours. All the materials tested were therefore treated in this way before the final measurements were made.

A slight modification of the arrangement described above is being employed in an attempt to determine the thermal conductivity of liquids.

The research has been carried out at the Wandsworth Technical Institute, and my thanks are due to the Principal and Governors of that Institute for their kindly interest in the experimental work.

ABSTRACT.

The thermal conductivities of a selection of non-metals at 20°C. and 100°C. have been determined by precisely the same method, and in the main the same apparatus as was recently employed by the author in the determination of the "Thermal conductivity of some of the rarer

* J. K. Clement and W. L. Egy, *loc. cit.*
† A. Eucken, *loc. cit.*

metals and alloys." The substances treated were in the form of accurately turned cylinders, placed in an enclosure at constant temperature. Measurements were made of the heat given (electrically) to one end of the cylinder, and of the temperatures of this end and of the enclosure, and the thermal conductivity deduced from a simple formula. A table of results is given, which agree well with those obtained by Lees and other observers in cases where a direct comparison is available.

DISCUSSION.

Dr. HARKER mentioned that the formula used by the author assumed that the isothermal surfaces in the specimen were plane. This would not be far wrong in the case of good conductors, but it might introduce serious error in the case of substances such as those now treated. He also doubted the validity of the assumption that "h" was proportional to V for any but very small differences of temperature. The results given in the Paper for firebrick were about 20 times lower than those given by other observers, including recent determinations of his own.

Mr. F. E. SMITH asked why the author put his second thermometer *inside* the enclosure maintained at a constant temperature. If the enclosing vessel is of metal, and water at constant temperature circulates, is it not better to take the temperature of the water? There must be a temperature gradient between the hot rod and the enclosing vessel. The second Pt thermometer lies on the gradient. Does the author assume the Pt thermometer to have the same temperature as the enclosing vessel? With regard to the temperature distribution at the hot end of the rod, he was at first of opinion that the author was in error, but afterwards found it difficult to make a definite statement owing to lack of data. He would, however, be glad if the author would give the solution of the following problem. Instead of a poor conducting rod, imagine a rod of an absolute insulator with regard to heat (*i.e.*, it is impossible to change its temperature) to be used. Then no heat can be given to the Pt thermometer from the end of this rod, and it appears that the temperature of the heater must be higher than before if the Pt thermometer has the same temperature. Will the author give his ideas as to the distribution of temperature in this case and that when the copper plug is substituted, in order to see whether it is likely that the conductivity of the rod would come out zero? He did not like the symbols used by the author, and thought it would have been simpler to give the energy rather than readings of E.M.F. and resistance.

Dr. RUSSELL said that it was important to know how much of the heat was lost by radiation and how much by convection currents. The author's emissivity constant seemed to include both. With ordinary copper wires suspended in air it was known that about 90 per cent. of the total heat emitted was taken away by convection currents, and only about 10 per cent. was lost by radiation. Newton showed that the convected heat carried away was proportional to the difference of temperature between the cooling body and the surrounding medium, and hence, so far as the convection loss was concerned, the author's assumption was justified. Stefan's law applied to the radiation loss, and, strictly speaking, the assumption that this loss was proportional to the difference of temperature was only permissible for small differences of temperature. Luckily, however, this loss was small, and he did not think that the accuracy of the results obtained was appreciably affected by the assumption.

The AUTHOR communicated the following reply: The points raised by Dr. Harker are, I believe, in each case due either to misapprehension or to misprints in the advance proofs. In the first place, the diameters of the specimens were 5 to 6 *millimetres* (not centimetres, as given in *one*

part of the Paper). The ratio of the sectional area to the thermal conductivity thus remains approximately the same as in the experiments on metals. This part of the criticism applies to practically all recent determinations of the thermal conductivity of metals where the dimensions of specimens are of the same order as in the present experiments. As Dr. Russell remarks, Newton's law of cooling is applicable to that part of the heat lost by convection, and also very nearly to that lost by radiation, provided the excess of temperature of the rod is not great. In the present research this excess of temperature was no greater than 10·6°C. Finally, the value of "k" in the case of firebrick is that given in the main table of results—viz., 0·0010. The value referred to by Dr. Harker is again a misprint. In reply to Mr. Smith, it is unsafe to put a bare platinum thermomcter in water or steam, and, in addition, the temperature to be measured (Pt_2) is that of the enclosure, which may be (especially at steam temperatures) slightly different from that of the "jacket." I feel certain there is little or no temperature gradient within the enclosing vessel, as the water equivalent of the latter is enormously greater than that of the rod, which is, in addition, never more than 10°C. higher than that of the enclosure. With regard to the problem proposed by Mr. Smith, there is no such thing as a perfect heat insulator. It may be safely assumed that the "hot" end of the rod acquires the temperature of the copper cylinder as given by the thermometer Pt_2. It might happen, however, that the rods were of such low conductivity that a smaller amount of heat is lost from it than from the end of the copper plug ($h.q.$V). This contingency is allowed for in the calculations, for the value of CE—C′E′ would then become negative. In the extreme case given by Mr. Smith, CE—C′E′ would be just equal to $h.q.$V, and would, therefore, give a zero value for H, and therefore also for the conductivity.

VII. *Ionisation.* *By* Sir J. J. Thomson, *O.M., F.R.S.*

By ionisation we mean the production in a medium, whether solid, liquid or gaseous, of particles charged with electricity which can be set in motion by an electric field and produce a current of electricity through the medium. Ionisation can be brought upon in many ways: By light, as when light falls on a metal surface, or when ultra-violet light of less than a certain wave-length passes through a gas; by Röntgen rays, which ionise solids and liquids, as well as gases, when they pass through them; by cathode rays, by positive rays, by heat as in the gases in a flame, by solution, by bubbling gases through liquids. In some cases possibly by chemical action, though in the great majority of cases the ionisation is due to the heat produced rather than to any special ionising effect due to chemical change. We know, for example, many cases of chemical action which are not accompanied by ionisation. The combination of hydrogen and chlorine is a conspicuous one, another is the dissociation of nickel carbonyl, which takes place at comparatively low temperatures, but is not accompanied by ionisation.

What is the nature of the carriers of these electric charges?

We must distinguish here between two different questions: (*a*) The nature of the carriers when they are first produced, and (*b*) when they have got into equilibrium with their surroundings. The study of the mobility of the ions, the speed at which they diffuse through the surrounding gas, or their velocity when acted on by given electric forces has thrown considerable light on the second question, the nature of the ions when in equilibrium. The most important properties brought to light by researches on the mobility of the ions are the following:—

1. The mobility of an ion depends only on the nature of the gas through which it is moving, and not upon the gas from which the ions are produced. Thus, if we ionise by Röntgen rays a mixture of iodine vapour and hydrogen gas, some of the ions will arise from the ionisation of the iodine and others from that of the hydrogen, the mobilities of the two kinds of iron through the mixture are the same.

2. Through a considerable range of pressures, the mobilities are inversely proportional to the pressure. In the case of the negative ions when the pressure is reduced below that due to a centimetre or so of mercury the mobility increases much

faster with diminution of pressure than this law indicates. Some recent experiments by Mr. Todd indicate that a similar result is true for positive ions, though the pressure at which the abnormality sets in is very much lower than for negative ions.

3. When the density of a gas remains constant the mobility of ions through it is independent of the temperature through a wide range of temperature.

4. The mobilities through several gases, such as hydrogen, argon, nitrogen, are very approximately inversely proportional to the square roots of their atomic weights; this is not true for easily condensed gases, and there are some exceptions to it for the more permanent ones.

5. The mobility does not depend on the charge on the ion—*i.e.*, it is the same for singly as for doubly charged ions.

Theories of the Mobility of Ions.

Expressions for the mobility of ions through gases have been worked out on two assumptions: (1) That the effects of collisions between ions and molecules are similar to those between two hard elastic spheres, and (2) that the effects produced by the molecules on the ions are the results of attractive forces between the atoms and molecules varying inversely as the fifth power of the distance between them; this law represents the force between a charged particle and an uncharged conductor when the distance between the two is large compared with the linear dimensions of the conductor. On the first theory I find, taking into account "persistence of velocities" (*see* Jean's "Dynamical Theory of Gases," p. 236), that R, the velocity of the ion under unit electric force, is given by the equation,

$$R=\frac{e}{N\pi\sigma^2\sqrt{M}\,.\,\Omega}\left(\frac{M_1+M_2}{M_1M_2}\right)^{\frac{1}{2}}\frac{5}{4}\left(\frac{M_2+3M_1}{M_2+5M_1}\right),$$

or

$$R=\frac{e}{N\pi\sigma^2\sqrt{M}\Omega}\left(\frac{M_1+M_2}{M_1M_2}\right)^{\frac{1}{2}}\frac{5M_2{}'+(M_1+3M_2)}{15M_2{}^2+10M_1M_2-M_1{}^2},$$

according as M_1 is greater or less than M_2. N is the number of molecules of the gas per unit volume, M the mass of a hydrogen molecule, Ω its average velocity at the temperature of the gas, M_1 the mass of an ion, M_2 that of a molecule, σ the sum of the radii of an ion and a molecule, and e the charge on an ion.

Let us first consider the variation of R with the masses of either the ion or molecule.

If $M_1 = M_2$,

$$R = \frac{e}{N\pi\sigma^2\sqrt{M}} \cdot \frac{5}{\Omega 6}\sqrt{\frac{2}{M_2}}.$$

If M_1 is large compared with M_2, then

$$R = \frac{e}{N\pi\sigma^2\sqrt{M}} \cdot \frac{3}{\Omega 4}\sqrt{\frac{1}{M_2}}.$$

Thus, if σ were unaltered, the mobility of an ion would become independent of its mass as soon as that became a considerable multiple of that of the molecule of the gas through which the ion was moving. The fact that the mobility of all ions through the same gas is the same implies on this theory that all the ions should be of the same size, and, what is still more remarkable, since in many gases the mobility of the ions varies inversely as the square root of the atomic weight, it follows from the expression for R that for all these gases σ should have the same value—*i.e.*, that in all these different gases the ions should be of the same size. From the values of the coefficients of mobility for positive ions I have calculated the relative values of σ for the positive ions in a number of gases; the results are given in the following table:—

Gas	σ^2
H_2	62·0
O_2	63·8
He	61·5
Cl_2	56·5
N_2O	69·0
CO_2	69·0
CO	90·0

These values are on an arbitrary scale; the absolute value of σ calculated by the same formula for hydrogen is $\frac{2}{\pi} \times 10^{-7}$cm. the value usually assigned to the diameter of a moelcule of hydrogen is $2{\cdot}1 \times 10^{-8}$cm. Thus the diameter of the cluster formed by the ion and a hydrogen molecule is about π times the diameter of a hydrogen molecule. Thus, though the ion is bigger than the molecule, it is not extravagantly so, and as the size of the ions in different gases tend to equality, the disproportion between the sizes of the ions and molecules would be less for the heavier and larger molecules than for those of hydrogen. In the case of easily condensed gases, such as ammonia gas, the vapours of alcohol and ether the size of the ions is considerably greater than for the more permanent gases.

According to the expression for R given above, the effect of

temperature on the mobility of the ion would depend upon the variation of $1/N\sigma^2\Omega$ with the temperature, or, when the gas is kept at constant volume, so that N is constant, of $1/\sigma^2\Omega$. But Ω varies as $\theta^{\frac{1}{2}}$ when θ is the absolute temperature, so that the temperature variation would be that of $1/\sigma^2\theta^{\frac{1}{2}}$. Now Phillips' experiments show that, through a considerable range of temperature, R for constant volume is independent of the temperature. This necessitates $\sigma^2\theta^{\frac{1}{2}}$ being constant. We may, perhaps, account for this in some such way as the following: The aggregates which form the ions are doubtless brought about by the condensation of the molecules under the attraction exerted by the electric charge at the nucleus of the ion. Statistical equilibrium will occur when the destructive effect of the collisions of the molecules against the aggregates is balanced by the condensation due to the electrical attraction, and the thermodynamics of this process shows that when this equilibrium occurs the work required to detach an uncharged molecule from the aggregate must be commensurate with the kinetic energy of a molecule of the gas—*i.e.*, must be proportional to the absolute temperature. If the uncharged molecule is regarded as analogous to a conducting sphere, and the electrical action of an ion as that due to a charge, e, at its centre, the work required to separate the uncharged molecule from the ion would be proportion to $\frac{e^2}{\sigma^4}.v$, when v is the volume of the uncharged molecule. In equilibrium this is proportional to θ; hence $\frac{e^2}{\sigma^4}$ is proportion to θ or σ^2 proportional to $e\theta^{-\frac{1}{2}}$. Thus $\theta^{\frac{1}{2}}\sigma^2$ would be constant, and therefore R for constant density independent of the temperature.

As on this view $\frac{e^2}{\sigma^4}v$ is proportional to the absolute temperature, e/σ^2, and therefore R, should vary as $v^{-\frac{1}{2}}$, and thus the velocity of the ions through different gases should vary as $\frac{1}{\sqrt{M_2 v}}$, and not as $\frac{1}{\sqrt{M_2}}$, which is what the experiments indicate.

If we suppose that from time to time the positive ion attracts a corpuscle and remains connected with it for a short interval, the connection being so loose that it can be broken up by collision with the molecules without the aid of any external agent, such as Röntgen rays; the mobility as determined by

many of the methods would be reduced by an amount depending on the proportion between the time the ion was free from the corpuscle to the time when it was in combination with it. This correction would depend on the intensity of ionisation, and would be greater when the ionisation was strong than when it was weak, and thus the mobility of the ions would depend upon the intensity of ionisation. No such effect has yet been discovered in gases at ordinary temperatures, and we conclude, therefore, that in these cases the correction is not important.

On the theory indicated above, $\frac{e}{\sigma^2}v$ depends only upon the temperature; thus σ^2 will be proportional to e, so that the mobility of the ion will not depend upon its charge. This is in agreement with experiment.

When the mass of the ion is very small compared with that of the molecules of the gas, as is the case when the negative ion is a corpuscle, the expression for R becomes

$$R=\frac{e}{N\pi\sigma_1^2\sqrt{M}\Omega}\sqrt{\frac{1}{M_1}},$$

where σ_1 is now the radius of a molecule of the gas. Thus the mobility on this view is increased in the proportion of $\sqrt{M_2/M_1}$ in consequence of the diminution of mass; also in the proportion of σ^2/σ_1^2 in consequence of the corpuscle being so much smaller than the ion. In gases such as air, hydrogen, oxygen the velocity of the negative ion is a little greater than that of the positive at pressures down to a centimetre or so of mercury. At lower pressures it becomes much greater than that of the positive ion, and the ratio of the mobility of the negative to the positive increases as the pressure diminishes. This suggests that at these lower pressures the negative ion is for a considerable fraction of its existence a free corpuscle, but at times gets attached to a molecule or aggregate of molecules of the gas, when it moves much more slowly. After a time it can get detached from this aggregate by causes other than the primary ionising agent. The negative ion thus alternates between a free corpuscle and a negatively charged aggregate, and the mobility will depend on the relative times it spends in these conditions.

In the cases of argon and nitrogen very carefully freed from all traces of oxygen or water vapour, Franck and Pohl have shown that even at atmospheric pressure the mobility of the negative ion is as much as 200 times that of the positive one,

showing that the negative corpuscles cannot attach themselves to molecules or aggregates of molecules of these gases. They have shown, too, that the presence of even a fraction of 1 per cent. of oxygen is sufficient to reduce the mobility of the negative ion to a value but little greater than that of the positive one.

Let us now consider for a moment another theory of the mobility of the ions, the theory which supposes that the motion of the ions through a gas is determined, not by collisions like those between elastic spheres, but by forces between the ions and the molecules varying inversely as the fifth power of the distance between them. Maxwell worked out the theory of diffusion for gases when the molecules repelled each other with forces varying inversely as the fifth power of the distance between them. We can easily adapt this theory to the case of an ion moving through molecules which it attracts with forces varying inversely as the fifth power of the distance. If we do this we find that R, the mobility, is expressed by the equation,

$$R = \frac{e}{NA\sqrt{\frac{M_1+M_2}{M_1M_2K}}},$$

where A is a constant, K the magnitude of the force between an ion and a molecule at unit distance, and, as before, e is the charge on an ion, M_1 and M_2 the masses of an ion and molecule respectively, and N the number of molecules per unit volume. It will be seen from this equation that as soon as M_1 becomes considerable with respect to M_2 the value of R depends but little on that of M_1. On this theory, then, if the ion is a cluster of molecules, the mobility will depend but little upon the number of molecules in the cluster, it does not in this case matter about the size of the cluster, as there is no factor corresponding to the σ^2 in the expression for R on the previous theory. Again, if K the measure of the force between an ion and a molecule, is independent of the temperature, R will, for a gas at constant volume, be independent of the temperature, a result indicated by Phillips' experiments. Again, if the force between the ion and the uncharged molecule is due to the charge on the ion, K will be proportional to e^2, so that R will be independent of the charge which is again in agreement with the facts.

We have seen that the mobility of the ions through several gases varies as $M_2^{-\frac{1}{2}}$; interpreted in the light of the preceding expression, this means that K is the same for all these gases.

This is a difficulty in the way of this theory. We should expect K to be proportional to the moment of the electrical doublet induced in the molecule by the charge on the ion when placed at unit distance from the molecule, but μ^2-1. if μ is the refractive index of the gas, is proportional to the moment of the electrical doublet induced in a molecule of the gas by a constant and uniform field of electric force; we should, therefore, expect K to be proportional to μ^2-1, and R to $((\mu^2-1)M_2)^{-\frac{1}{2}}$, and not to $M_2^{-\frac{1}{2}}$. There is no doubt, however, that in many gases R is much more nearly proportional to $M_2^{-\frac{1}{2}}$ than to $\{(\mu^2-1)M_2\}^{-\frac{1}{2}}$.

Let us now consider what on this theory would be the mobility of a negative corpuscle; in this case M_1 is very small compared with M_2, so that the formula becomes

$$R=\frac{e}{NA}\sqrt{\frac{1}{M_1K}},$$

while the mobility of an ion through the same gas is

$$\frac{e}{NA}\sqrt{\frac{1}{M_2K}},$$

if the force between the ion and the molecule were like that between a point charge of electricity and a conducting sphere, or a sphere whose specific induction capacity was greater than that of the surrounding space, K would be the same for the ion and the corpuscle, so that the mobility of the corpuscle would be $\sqrt{M_2/M_1}$ times that of the ion; in the case of nitrogen, when $M_2/M_1=28\times1{,}780$, the mobility of the corpuscle would be about 220 times that of the ion, so that at 760 mm. pressure and 15°C. the mobility of the corpuscle would be about 200 (cm./sec.) (volt/cm.). This is of the order found by Franck and Pohl for the mobility of the negative carriers in nitrogen carefully freed from oxygen. On the other theory of mobility, the mobility of a corpuscle would be more than four times this; for the mobility of the corpuscle is $\frac{\sigma^2}{\sigma_1^2}\sqrt{\frac{M_2}{M_1}}$, where σ is the sum of the radii of an ion and a molecule, and σ_1 the radius of a molecule, and σ is more than double σ_1.

Again, in those gases like hydrogen, helium, oxygen, nitrogen and argon, where R is proportional to $M_2^{-\frac{1}{2}}$, K must be constant, so that in all these gases the mobility of the corpuscle must be the same, a result which, as far as I know, has not yet been tested.

For gases such as the vapours of alcohol, water or ether, which have abnormally large specific inductive capacities in comparison with their refractive indices, I have given reasons for thinking ("Phil. Mag.," May, 1914) that some of the atoms in the molecule are positively, others negatively charged, so that the forces between these molecules and a charge of electricity would tend to vary rather as the inverse cube than the inverse fifth power of the distance. To these, therefore, the preceding theory would not apply, and as a matter of fact, the mobilities in these vapours are considerably less than those in the permanent gases.

The Communication of Electricity from a Gas to a Metal.

The study of the constitution and properties of ions leads to results which have an important bearing on the question of the communication of a charge of electricity from gas to a metallic electrode immersed in it. We have seen that in the inert gases, such as nitrogen and argon, the corpuscles in the gas do not attach themselves to the molecules. When they come into collision with these molecules they rebound in their original condition; the impact of a corpuscle against a neutral molecule or atom does not result in the charging of the atom. Again, to deprive a positively electrified ion of its charge a corpuscle must attach itself to the ion. This corpuscle must come from somewhere, and to detach a corpuscle requires the expenditure of a finite amount of energy. The study of positive rays again shows that to deprive some negatively electrified atoms of their charge, those of oxygen, hydrogen or carbon, for example, requires the expenditure of a considerable amount of work. We thus see many examples where the mere contact between the electrified body and a neutral one does not result in the transference of the electrification.

Let us apply these considerations to the case of a metal electrode immersed in an ionised gas. From what we have seen, it is unlike that the mere hitting of the ions against the metal will result in the transference of the charge from the gas to the metal; to effect this transference, in many cases at least, will require the expenditure of a finite amount of work. Where is this work to come from? The most obvious source of work of this kind is to suppose that if the metal is acting as an anode, negative ions or corpuscles accumulate in its neighbourhood, forming, with the surface of the anode, two layers of electrification, with a finite difference of potential between them. If V

were this difference of potential then there would be an amount of work, Ve, available to transfer a charge, e, from the gas to the metal. If we were the work required, for example, to transfer the negative charge on a negatively electrified oxygen ion, the transference of the charge from oxygen gas to the anode would not occur unless there was a difference of potential at least equal to w between the anode and the gas. Thus, we should expect to find considerable difference of potential between metal electrodes and the surrounding gas. This consideration is one that should be borne in mind in interpreting the results of some experiments on the discharge through gases. For example, it is often assumed that the potential in the interstices of a piece of wire gauze is the same as that to which the gauze is raised by its metallic connection with a voltaic battery. If double layers are formed round the wire of the gauze there may be a considerable difference of potential between the wires and the open spaces, so that the establishment of a retarding potential V between the wires of two gauzed screens may not be sufficient to stop all the rays with energy corresponding to V from passing through the screens. An example, which is interesting historically, of this effect is afforded by earlier investigations on the cathode rays. It was at first thought that these could not be charged particles, because they were not deflected by an electrostatic field, even when the charged plates used to deflect the rays were placed inside the vacuum tube. This want of deflection was due to the fall of potential between the plate being used up in the formation of double layers at their surface, so that the greater part of the space between the plates was comparatively free from electric force. When the pressure in the vacuum tube is very much reduced, the effect of these double layers is diminished, though even at the lower pressures which are now used for these experiments it is necessary to be alive to the fact that double layers may still be present, and that the electric force to which the cathode particles are exposed is not necessarily equal to the potential difference between the plates, divided by the distance by which they are separated. We should expect these double layers to be especially conspicuous in strongly electro-negative gases like oxygen and chlorine, which require a large amount of work to deprive them of their negative charge.

A conspicuous instance of the formation of a double layer is that at the anode in a vacuum tube through which a current of electricity is passing. This produces a change of potential

close to the anode amounting in many cases to more than 20 volts, and it occurs so abruptly that it has hitherto been impossible to reach a point so close to the anode that the difference of potential between the anode and the point is less than this value. The experiments of Skinner show that it varies somewhat with the metal of the electrode, though the variations are not large and it does not vary much with the current density at the cathode. It is exceptionally large in chlorine. It may be regarded as the mechanism by which the necessary work is supplied (1) to get the negative charge from the ions in the neighbourhood, and (2) in some cases to liberate positive ions from the anode.

Experiments on the mobility, however, in the main give us information as to the size of the ion when it is in equilibrium with its surroundings; they give us but little information as to what is the nature of the ions when they are first formed, and therefore give but little information as to the processes by which ionisation is effected. It is, however, the mechanism of this process to which I wish to direct your attention rather than to its final product. Many of the points in this process are obscure and the investigation of them difficult and complicated; but the question is one of such importance that I think no apology is needed for bringing it before you this evening.

One point in the mechanism of ionisation seems quite clear—that is, that in, at any rate, the vast majority of cases of ionisation of gases, one part of the process of ionisation is the liberation of a corpuscle or electron. We know this, because, at very low pressures, and in the case of inert gases like argon even at high pressures, the measurement of the mobility and diffusion of the negative ions shows that these are many hundred times greater than the corresponding quantities for the positive ions. At higher pressures, on the other hand, the difference, except in the case of the inert gases, is comparatively small, for here the surrounding molecules have had time to condense round the electron and load it up before it is finally driven out of the gas. Thus, the process of ionisation involves the liberation of a corpuscle from the molecule, and is not primarily the splitting up of a diatomic molecule into two oppositely charged atoms.

One part of the process of ionisation is thus the ejection of a corpuscle from the molecule of the gas which is ionised. The violence with which the corpuscle is ejected depends to a great extent on the way the ionisation is produced, and the study of

the way the velocity of the corpuscle on its departure from the atom varies with the type of the ionising agent gives some very interesting results. When a gas is ionised by Röntgen rays the corpuscles are expelled from the molecules with very high velocities; in fact, the experiments made on this subject point to the conclusion that the velocity with which the corpuscles are expelled does not differ greatly, if at all, from that of the cathode rays, which by their impact against the anti-cathode generated the Röntgen rays which ionised the gas. The existence of the high-speed corpuscles coming out of molecules exposed to Röntgen rays is clearly shown by C. T. R. Wilson's photographs of the distribution of the ions produced when a gas is exposed to Röntgen rays. We see that the drops of water which mark the situation of the ions lie along a series of detached fine curves, which mark the track of the high-speed corpuscles emitted when the gas is exposed to the rays. These high-speed corpuscles ionise the gas as they pass through it, each one of them producing a large number of ions; thus the greater part—in fact, practically the whole—of the ionisation in gases under Röntgen rays is produced, not directly by the rays, but indirectly by the rapidly-moving corpuscles which they eject.

The velocity of these fast corpuscles does not depend upon the nature of the gas, nor on the intensity of the rays, but only upon the quality—the hardness of the rays. The harder the rays the greater the speed of the corpuscles. A similar result holds when ultra-violet light is used instead of Röntgen rays. The velocity of the negative particles expelled from a metal plate under the influence of ultra-violet light does not depend upon the intensity of the light, but merely upon its wave-length. The experiments made by Mr. Hughes point to the conclusion that the velocity with which they leave the molecule, as distinguished from the velocity with which they leave the plate, does not depend on the kind of molecule, but merely on the wave-length of the light.

Thus the velocity of the particles expelled by means of radiation, whether this be light or Röntgen radiation, depends solely upon the quality of the radiation, and when this radiation has, as is the case with monochromatic light or characteristic Röntgen radiation, a definite frequency, the energy with which the corpuscles are expelled from the molecule is proportional to the frequency of the radiation. This is a result which might be expected from the "quantum" theory. The experimental results hitherto obtained seem on the whole to confirm this

result, though it must, I think, be confessed that the direct experimental evidence in favour of the strict proportionality is too meagre to enable us to maintain that it is a direct result of experiment. There are many points on which we have no direct experimental evidence; for example, the energy measured is the maximum energy of the ejected particles; we do not know whether the energy of a corpuscle ejected from a molecule by monochromatic light has a constant value or whether it can assume a considerable range of values stopping at a definite maximum. The discrepancy between different experimenters on this subject is considerable, and it is not impossible that the limit they obtain for the maximum energy may depend on the sensitiveness of the recording instruments and the excellence of the insulation. In spite of this, it is abundantly proved, however, that the speed of the emitted corpuscles varies very rapidly with the nature of the ionising agent.

The energy acquired by the ejected particles under the influence of radiation is so great that it is very difficult to reconcile it with the idea that the energy in the radiation is uniformly distributed through space. On the assumption that the energy is uniformly distributed, we can calculate the maximum electric force in a beam of light of known intensity, and we find, after giving the greatest possible value to any resonance effect, that to account for the velocity of the ejected particles we should require uninterrupted trains of waves millions of wave-lengths long. This difficulty is overcome if we adopt a modification of an idea that I put forward some time ago. At that time it was thought that the total number of ions produced when a given beam of Röntgen rays was absorbed in a gas was the same for all gases, and I suggested that the Röntgen beam was made up of units, and that whenever an ion was liberated one of these units was absorbed. Subsequent experiments, however, showed that the number of ions produced by the absorption of a given beam depended to an appreciable extent upon the nature of the gas and also that the production of the ions was in the main due to the high-speed corpuscles ejected by the rays, and not to the rays themselves. Recent experiments by Barkla and Philpotts have shown, however, that, though the number of ions produced varies with the gas which absorbs the radiation, the number of the high-speed particles ejected is invariable, so that we may assign to these high-speed particles the rôle which I previously assigned to the ions, and suppose that the liberation of a high-

speed particle from any gas whatever involves the destruction of one unit of radiation. We may picture to ourselves the process as being somewhat of the following type: We suppose that radiation, whether ordinary light or Röntgen radiation, is made up of units, and that these units are closed tubes of electric force, each tube possessing a definite amount of energy depending upon the type of radiation. Suppose, now, that one of these closed tubes, C, travelling through space passes near to a molecule of a gas, and that in this molecule there is a tube of force passing between a corpuscle, A, and the equivalent positive charge B. If, now, the part of C which passes nearest to AB is in the opposite direction to AB, the two tubes will attract each other, C getting pulled down to AB and AB up to C, as in Fig. 1 (A). If this process goes on until the tubes come into contact, as in Fig. 1 (B), fissure will take place as in Fig. 1 (C), and the two tubes C and AB will be replaced by a long tube with its ends anchored at A and B. Thus the closed

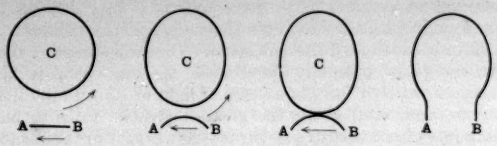


FIG. 1. FIG. 1A. FIG. 1B. FIG. 1C.

free tube which could travel freely through space and formed a unit of the radiation, has ceased to exist, and has been replaced by an anchored tube; thus one unit of radiation has ceased to exist. The anchored tube is in a state of tension, and therefore pulling at the corpuscle. This pull first detaches the corpuscle from the molecule, and then increases its velocity, so that the energy which was in the tube gets communicated to the corpuscle, and all but the work required to detach the corpuscle from the molecule appears as kinetic energy in the corpuscle. If the work required to detach the corpuscle is but a fraction of the kinetic energy it acquires, that kinetic energy will be approximately equal to the energy originally in the ring C, the unit of radiation, and so will practically be independent of the substance from which the corpuscle is liberated.

Let us now turn to another ionising agent—rapidly moving electrified particles. Let us first take the case of negatively electrified particles, cathode rays. We find that these cannot

ionise a gas unless they possess more than a certain amount of energy. This amount can conveniently be expressed in terms of the voltage through which the atomic charge must fall to acquire the requisite amount of energy. This voltage, according to Franck and Hertz, is as follows :—

Hg	N	O	H	Ar	Ne	He
4·9	7·5	9	11	12	16	20·5 volts.

There is no apparent connection between these quantities and the chemical properties of the atom, nor, what is perhaps still more striking, with their power of acquiring a negative charge as contrasted with that of losing it. For example, neither the atoms of nitrogen and helium, the elements at the ends of these series, acquire negative charges in the positive rays, while those of oxygen and hydrogen readily do so ; again, argon and neon, which are also in intermediate positions, do not.

When cathode rays with more than the ionising energy move through these gases they expel corpuscles from some of

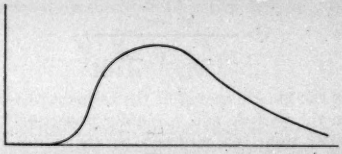


Fig. 2.

the molecules against which they strike. The numerous experiments which have been made on the velocity of corpuscles expelled in this way have led to some extremely interesting results.

It has been found that the velocity of these corpuscles is practically independent of the velocity of the cathode rays which drive them out of the molecules, and that the maximum energy is not above that acquired by the fall of the atomic charge through about 10 volts ; thus, even though the cathode particles have speeds corresponding to that represented by thousands of volts, the corpuscles they eject, or, at any rate, the great majority of them, move little if at all faster than those liberated by rays which have a speed represented by 20 or 30 volts. The number of the corpuscles liberated by a cathode particle when moving through 1 cm. of gas at a definite pres-

sure is represented by a curve of the type shown in Fig. 2. The ionisation does not begin until the cathode rays have a certain amount of energy. It then increases rapidly, attains a maximum which, in the cases investigated by Kossel, was when the cathode rays had energy corresponding to about 300 volts, then diminishes slowly at first, but ultimately inversely as the energy of the cathode particle. A curve almost identical with this has been shown by Gehrts ("Ann. der Phys., XXXVI., p. 1003) to represent the secondary cathodic radiation excited when a plate of metal is bombarded by cathode rays of different speeds. Here, as in the case of gases, the speed of the secondary rays does not seem to depend much upon that of the primary. The most obvious explanation of ionisation by cathode rays is that a corpuscle in the atom acquires from a cathode particle moving past it sufficient energy to enable it to escape from the atom. The energy Q communicated to a particle at rest of mass M by a moving particle of mass m, starting with the velocity V, is given by the equation,

$$Q=\frac{\frac{4mM}{(M+m)^2}T}{1+\frac{d^2\cdot 4}{e^2E^2}T^2\left(\frac{M}{m+M}\right)^2},$$

where T is the kinetic energy of the cathode particle, e, E the charges on the particle and corpuscle respectively, and d the perpendicular from the corpuscle on the initial path of the particle. If the corpuscle requires an amount of energy, W, to free it from the atom, the particle will only eject the corpuscle when

$$d^2<\left\{\frac{4mM}{(M+m)^2}\frac{T}{W}-1\right\}\frac{e^2E^2}{4T^2\left(\frac{M}{m+M}\right)^2}.$$

The number of ions per centimetre of path of the cathode particle will be proportional to the limiting value of d^2, and thus, regarded as a function of T, will vary as

$$\frac{1}{T^2}\left\{\frac{4mM}{(M+m)^2}\frac{T}{W}-1\right\}.$$

Thus ionisation will not begin until

$$T=\frac{(M+m)^2}{4mM}W,$$

and will be a maximum when T has twice this value; for large

values of T the ionisation will be inversely proportional to T. When we compare this with the results found by experiment we see that, though some of the more striking phenomena are accounted for by the theory, there are others which are not in accordance with the theory in this form. One of these is the fact that the energy of the primary cathode rays when the ionisation is a maximum is much greater than twice the energy when ionisation first begins. This, however, could be accounted for by supposing that of the corpuscles in the atom some require more energy than others to eject them from the atom. There may be one group that requires W_1 units of energy to escape, another W_2 and so on; thus ionisation will begin when the primary cathode rays have energy equal to W_1, and though the detachment of this group will be a maximum when the energy of the cathode rays is $2W_1$, yet, as the energy of the cathode rays increases, the other groups of corpuscles may begin to be ejected, with the result that the energy of maximum ionisation is raised. If there are n_1 corpuscles which require energy W_1 to eject them, n_2 requiring W_2, we are putting $M=m$ in the equation that the ionisation will be a maximum when

$$T=\frac{2\Sigma n}{\Sigma\frac{n}{W}},$$

and will begin when $T=W_1$; in this case the energy for maximum ionisation cannot be calculated from that for initial ionisation. To avoid subsidiary maxima, of which the curves connecting the ionisation with T show no indication, W_2 must be $<2W_1$ and $W_3<2(n_1+n_2)\Big/\left(\frac{n_1}{W_1}+\frac{n_2}{W_2}\right)$, and so on. Another difficulty in the way of this theory is that, as Pawlow has shown, positive rays can ionise a gas when their energy is as small if not smaller than that required by cathode particles for this purpose. In the case of a positive ray colliding with a corpuscle the masses of the particles are so different that only a small fraction of the energy of the moving particle can be transferred to the corpuscle; in fact, the equation shows that when M is large compared with m, ionisation will not occur unless $T=\frac{M}{4m}.W_1$, while in the case of the collision between two corpuscles it might occur if $T=W_1$, thus if W_1 were 11 volts ionisation ought not to be produced

by positive rays unless their energy were at least $1\,700 \times 11/4$, or 4 600 volts, whereas Pawlow has shown it occurs with positive rays when T is less than 11 volts. Another discrepancy is that if the corpuscles were knocked out by direct collision we should expect to find the maximum velocity of ejection dependent on the speed of the particle ejecting them, whereas the experiments seem to prove that there is no such effect.

The result seems to suggest that the ejection of a corpuscle by a cathode particle or a positive ray is not the result of these colliding directly with the corpuscle, but results rather from the effect of the collision with the atom regarded as a whole. This collision produces a kind of convulsion in the atom, the effect of which is to eject the corpuscle from a position in which it was in stable equilibrium under the forces inside the atom. When driven out of this position it comes under the action of repulsive forces which eject it from the atom and the velocity of projection will depend upon the forces inside the atom, and not upon the speed of the particle striking against it. We should expect this speed to be, on this view, the same whether the atom were struck by a positive particle or by a cathode ray. It is some confirmation of this view that the maximum number of ions produced by a cathode particle in its path through a gas is approximately equal to the number of collisions the particle would make with the atoms of the gas, if the atoms were of the dimensions usually assigned to them on the kinetic theory of gases. It would, however, seem likely that very rapid cathode particles which could penetrate right through the atom might be able to eject from it corpuscles with a greater velocity than that corresponding to 10 volts or so, which seems to be the limit for those ejected by more slowly moving cathode rays. The corpuscles ejected at a high speed must, however, be at most a very small fraction of the whole number ejected, for the photographs of the paths of the β particles taken by Mr. C. I. R. Wilson show that these paths are very rarely forked, as they would be if the atoms struck by the β particles gave out corpuscles moving at a speed comparable with that of the primary ones. Such particles would be able to travel a finite distance through the air, and would cause the ions produced by a β ray to be distributed somewhat as in Fig. 3. The actual photographs of the ions show, however, few traces of such branches, and indicate that the corpuscles projected from the atom have not sufficient speed to carry them through at an appreciable

distance in gases at atmospheric pressure. This result raises some very interesting questions as to the production of Röntgen radiation by the impact of cathode rays. If we take the situation in air as revealed by the Wilson photographs to be typical of that occurring when cathode particles move through a collection of molecules of any kind, the cathode particle when it strikes against an obstacle dislodges from the atoms through which it passes slow secondary cathode rays, losing energy by this process in a more or less gradual manner, the amount of energy lost at each collision being but a small fraction of the energy of the cathode particle. It is hardly likely that encounters of this kind can give rise to Röntgen rays able to liberate corpuscles possessing a velocity comparable with that of the primary cathode particle. It seems to me more probable that the hard Röntgen rays produced by the impact of the cathode particles against the anti-cathode result from a process the inverse of that by which we supposed Röntgen rays to generate high-speed cathode rays. Let us suppose that at the stage in this process when the cathode

Fig. 3.

particle had acquired its final velocity its velocity were reversed, so that it retraced its path, and the whole process was reversed. The result would be that we should end by getting to the state which existed before the Röntgen ray struck the atom—*i.e.*, we should get the closed tube of electric force which on this view constitutes a Röntgen and the cathode particle that was originally free would be joined up in an atom with an equivalent charge of positive electricity. The cathode particle would cease to be a cathode particle and would be imprisoned in an atom, and thus the generation of a pulse of hard Röntgen radiation would be the swan's song of the cathode particle, the birth of a Röntgen ray involving the death of a cathode particle. The hardness of the Röntgen rays would depend upon the energy of the cathode particle; thus, those particles which got imprisoned before they had liberated many ions would produce harder Röntgen rays than those which only did

so after losing a large part of their energy by ionising the atoms through which they pass. Mr. Wilson's photographs show that most of the cathode particles go on until they produce considerably more ions per unit length at the end of their path than they did at the beginning, showing that at the end their energy is considerably smaller than it was at the beginning. Thus, a large number of cathode particles remain free until they have lost a considerable part of their energy, as these give rise to softer Röntgen rays than those which have not lost energy, we should expect that the Röntgen rays would be very far from homogeneous, and would contain a considerable proportion of comparatively soft rays.

There should also be another kind of radiation produced by the impact of cathode rays. We have seen that these expel from the atoms corpuscles with a velocity not exceeding that due to 10 or 11 volts. Now the sudden starting of these 10 volt corpuscles ought to produce radiation, which, if Planck's view as to the relation between frequency and energy were to apply to this case, would be in the Schumann region. As the speed of these particles does not depend much on the speed of the primary cathode rays or the nature of the substance, we should expect to get a radiation of much the same type whatever the speed of the cathode rays or the material they struck against. I am not aware that any invariable radiation of this kind has ever been observed. It would not be likely to be able to ionise such gases as hydrogen or oxygen, which have ionising potentials of more than 10 volts. It might, however, be able to ionise mercury vapour, for which only 5 volts are required.

In many respects the ionisation of gases by the impact of cathode particles shows the same characteristics as the emission of secondary cathodic radiation when a metal plate is bombarded by cathode rays. This effect has been studied by Gehrts ("Ann. der Phys.," 36, p. 995) and Campbell ("Phil. Mag.," Aug., 1914). There is no secondary cathodic radiation until the velocity of the primary rays exceeds 11 volts. It then increases rapidly with the speed of the positive rays, and attains a maximum when the speed corresponds to about 300 volts; it then diminishes as the voltage increases. The velocity of the secondary rays is less than that due to 11 volts, and does not depend on the velocity of the primary rays, nor the nature of the material against which they strike. In all these respects the effects are analogous to those obtained when a gas

is ionised by cathode rays, and indeed it seems clear that the physical processes going on in the two cases are identical—viz., the ionisation of the atoms. In one case the atoms ionised are those of a gas, and in the other those of a metal or, perhaps, gas absorbed in the metal. The fact that the secondary radiation does not begin until the energy in the primary rays have the ionising potential for hydrogen suggests, as Campbell has pointed out, that the secondary radiation may come from a film of hydrogen condensed on the surface of the metal, and that the curves obtained may really be those for the ionisation of hydrogen and not for the atoms of the metal. Particles at a higher speed might be expected to penetrate the layer and reach the metal. It would be interesting to see whether there is the same connection between ionisation of gases and secondary cathodic radiation at high speeds as there is at low. The rate at which both the secondary cathodic radiation and the ionisation of a gas increases with the energy of the incident cathode rays is very rapid when this energy corresponds to about 11 volts, the maximum energy required to ionise the gas. This ionisation may be regarded as the expulsion from the atom of the corpuscles which are most easily ejected; there are probably in the atom other corpuscles, more deeply seated which demand larger amounts of energy for their expulsion. In this case we should expect to find another stage, at which the ionisation would increase rapidly with the energy of the incident cathode rays. No experiments, as far as I am aware, have been published which give indications of such a stage until the energy of the cathode rays is sufficient to excite the characteristic radiation of the metal against which they strike. The experiments do not, however, extend over a very wide range of velocities of the primary rays. In Gehrts' experiments, for example, the maximum energy of the primary cathode rays was only that corresponding to about 600 volts. I am at present engaged with experiments on a point somewhat similar to this. Cathode rays, even when their energy does not exceed 50 or 60 volts, produce, when they strike against a target, appreciable amounts of very soft Röntgen radiation, and the experiments I am making are measurements of the amounts of this radiation when metals are bombarded with cathode rays of different speeds. If the atom contains a series of rings of electrons, then for each ring we should expect to get a special type of characteristic Röntgen radiation. This ought to make itself apparent by an abnormal

increase in the radiation produced by the incident cathode rays when their energy passes through the values required to detach a corpuscle from any one of the rings; thus, if we can detect these increases we ought to be able to measure the number of rings in the atoms of the different chemical elements. The experiments are not yet concluded, but I hope on a future occasion to bring the results before the Society. I may mention, in passing, that the impact of positive rays, as well as of cathode rays, against a plate of metal gives rise to a very soft type of Röntgen radiation.

A very important point about the velocity of the slow corpuscles emitted when substances are bombarded by cathode rays is that apparently it does not depend on the nature of the substance bombarded. It is to be noticed, however, that these substances have nearly always been in the solid state and liable to be coated with films of gas; the corpuscles may come from these films and not from the metal, so that the evidence on this point is not conclusive. More to the purpose would be measurements of the velocities of the corpuscles from different gases, especially from those which, like helium and nitrogen, require very different amounts of energy to ionise them.

Some ionisation seems to be produced by positive particles when their energy is less than that required by cathode rays for ionisation. When once the cathode rays, however, do begin to ionise they produce far more ions than positive particles of equal energy. Another case when we have ionisation produced by the impact of bodies of molecular dimensions is that of hot gases. In a gas at the temperature of 2,000°C. the average kinetic energy of the molecules is that due to the fall of the ohmic charge through about 0·25 volt; hence, the number of molecules which possessed energy corresponding to 11 volts would be a fraction of the order ε^{-44} of the whole number of molecules; a gas in which the ions were in this proportion to the molecules would have quite measurable conductivity. If, however, the hot gas were helium, which, according to Franck and Pohl, requires about 22 volts to ionise it, the particles possessing this amount of energy would only be ε^{-88} of the whole number, and this fraction of ions would not possess appreciable conductivity. It would be interesting to see if the conductivity of hot helium was very much less than that of hot hydrogen. The greater part of the ionisation of hot gases occurs at the junction of the gas with hot solids, and it

seems reasonable to suppose that gas condensed in a layer on the surface of a conductor would be more easily ionised than the gas in a free state. It would seem, however, from recent experiments by Campbell that a platinum plate, probably coated with a layer of hydrogen is not more easily ionised by cathode rays than free hydrogen.

We know that one part of the process of ionisation consists in the detachment of a corpuscle, but what happens to the rest of the molecule from which the corpuscle is detached? In the case of fast cathode rays we must remember that we have far more energy available than that which is usually supposed to be required for the dissociation of the molecule into atoms. When we study by the positive ray method the ions produced in a gas through which the electric discharge is passing we always find charged atoms of the gas through which the discharge is passing often, though not invariably, accompanied by charged molecules of the same gas. In this case, however, we have moving positive particles as well as cathode rays, and it might be thought that the charged atoms were produced by the positive rays, the charged molecules by the cathode rays. To test this point I used a discharge tube whose cathode consisted of a spiral of tungsten wires, the positive rays passing through the spiral. By heating the tungsten wire to a white heat it could be made to give out a large output of negative corpuscles, and so when the spiral was hot a larger proportion of the current would be carried by the negative particles than when it was cold. If, then, the cathode rays only produced charged molecules and not atoms the ratio of the charged molecules to the charged atoms would be greater when it was hot than when it was cold. I found, however, that the proportion of charged molecules of hydrogen to charged atoms was less for the hot tungsten than the cold, and hence I conclude that ionisation by fast cathode rays is frequently accompanied by the dissociation of the molecule into atoms. Whether this is so or not, when Röntgen rays expel high-speed corpuscles from gases through which they pass is a question on which, as far as I know, we have at present no evidence.

ABSTRACT.

Ionisation is the process by which ions—particles charged with electricity—are produced in a solid, liquid or gas. This address is confined to the consideration of ions in gases and deals with two questions—(1) the nature of the ions, (2) the process by which they are

produced. The evidence as to the nature of ions is derived from experiments on their mobility which have shown :—

α. The mobility of a positive ion depends only on the gas through which the ions are moving and not on the nature of the gas out of which the ions are formed.

β. The mobility of the ions in a gas at constant density is independent of the temperature.

γ. There are a considerable number of gases in which the mobility is very approximately inversely proportional to the square root of the density of the gas.

δ. In the case of negative ions there is an abnormal increase of mobility when the pressure is reduced below a certain value, and there is some evidence that this is also the case for positive ions at very low pressures.

Two theories of the mobilities of ions were considered, one founded on the view that the action between ions and molecules is analogous to impacts between hard elastic spheres, the other on Maxwell's theory of forces between ions and molecules varying inversely as the fifth power of the distance between them.

It is shown that α follows from the second theory provided the ion is a cluster whose mass is considerably greater than that of a molecule of the gas through which it is moving. On this supposition it follows also from the first theory if we suppose in addition that all ions in a given gas are of the same size.

(β) follows at once from the second theory ; to explain it on the first theory we must suppose that the size of the ion varies with the temperature in a definite way. The necessary relation can be deduced from thermodynamical principles if we suppose that the force between an ion and a molecule is analogous to that between a charged point and a sphere which is either a conductor of electricity or has a high specific inductive capacity.

(γ) requires on theory one that the ions in these gases should be of the same size, on theory two that the molecules of these gases should exert the same force on a charged point at a given distance.

(δ) follows on either theory if the ion dissociates at low pressures so that free corpuscles are present in the gas.

With regard to the process of ionisation, the first stage of this in the vast majority of cases consists in the detachment of an electron or corpuscle. Evidence as to the method by which this takes place is afforded by determinations of the velocity with which the electrons are ejected from the body.

In the case of ionisation by light or Röntgen rays, this velocity depends primarily on the wave-length of the radiation, not upon its intensity, nor, at any rate to any great extent, on the nature of the molecule from which the corpuscle is ejected. This velocity is far greater, even when every allowance is made for resonance, than can be accounted for if we suppose that the energy of the light is uniformly distributed, and that the corpuscle acquires its velocity by the action on it of the electric force in the wave. Another explanation not open to these objections was put forward.

When ionisation is due to the action of moving electrified particles, whether positive or negative, the results are quite different. The

velocity of the ejected particles (δ-rays, as they are sometimes called) does not seem to vary much, if at all, with the velocity of the particles which eject them, and is of the same order whether these particles are positive rays or the much swifter cathode rays. The method of ejection by the impact of such particles was considered, and suggestions as to a possible theory and some of its consequences thrown out.

Dr. R. T. GLAZEBROOK, in moving a vote of thanks to the President, said that one expected, when listening to Sir J. J. Thomson, to be carried to the bounds of knowledge by one who had taken no small part in the extension of these bounds, and he was confident that none of those present had been disappointed in that expectation. He asked the President, on behalf of the Society, to allow his address to be published in full in the "Proceedings."

Dr. C. CHREE seconded the vote of thanks, which was carried with enthusiasm.

VIII. *Exhibition and Description of some Apparatus for Class Work in Practical Physics. By Dr.* G. F. C. SEARLE, *F.R.S., University Lecturer in Experimental Physics, Cambridge.*

DR. SEARLE exhibited and described the apparatus used for the following experiments in his class at the Cavendish Laboratory, Cambridge. The methods are briefly described in the following abstracts; the references indicate where fuller information may be found. In a few cases the complete Papers have not been published at the date of publication of these abstracts, but it is hoped that they will be published during the year 1915. The titles of these Papers are distinguished by an asterisk (*).

1. *An Experiment on the Harmonic Motion of a Rigid Body.** The experiment is designed to illustrate the formula

$$T=2\pi\sqrt{K/\mu},$$

where T seconds is the periodic time of a rigid body of moment of inertia K grm.cm.^{2} which is suspended from a fixed support by a torsion wire exerting a restoring couple of μ dyne-cm. per radian. When μ is to be found, a vertical cylinder is attached to the lower end of the wire and two threads are wound round the cylinder. These threads pass over ball-bearing pulleys and support known masses. The angular deflexion of the cylinder caused by the masses is measured with a goniometer by the method of auto-collimation. An inertia bar of known moment of inertia K is then substituted for the cylinder and the periodic time is observed. ("Proc." Cambridge Phil. Soc., Vol. XVIII., p. 31; also "Experimental Harmonic Motion," Cambridge University Press, 1915. This small manual is similar in character to the author's "Experimental Elasticity.")

2. *Determination of Gravity by a Rigid Pendulum.** The periodic time, T, of a rigid pendulum vibrating about a horizontal axis (knife edge) is given by

$$T=2\pi\sqrt{K/Mgh}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where K is the moment of inertia of the pendulum about its

axis of suspension, h is the distance of the centre of gravity from that axis and M is the mass of the pendulum. Since K cannot be satisfactorily calculated from the mass and dimensions of the pendulum, K must be found in some other way. By the theorem of parallel axes,

$$K=K_0+Mh^2, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

where K_0 is the moment of inertia of the pendulum about an axis through its centre of gravity parallel to the knife edge.

To determine K_0, the pendulum is suspended from a torsion wire by aid of a simple stirrup so that the rod of the pendulum is horizontal. The periodic time, T_0, of the torsional vibrations is found. An inertia bar of calculable moment of inertia K_1 is then attached to the torsion wire in place of the stirrup and pendulum and the periodic time, T_1, is observed. Then

$$K_0=K_1T_0^2/T_1^2.$$

This value of K_0 is used in (2) and so K is found. The distance h is found by balancing the pendulum on a fulcrum. (Searle, "Experimental Harmonic Motion," Cambridge University Press, 1915.)

3. *Determination of Poisson's Ratio for Indiarubber.**

Let the length of a tube of circular section be l, let its internal radius be r, and let its internal volume be v, so that $v=\pi lr^2$. Let the length be slightly increased so that l becomes l', r becomes r' and v becomes v'. Then, if σ is Poisson's ratio,

$$\sigma=\tfrac{1}{2}\left\{1-\frac{l}{v}\cdot\frac{v'-v}{l'-l}\right\}.$$

For infinitesimal changes we have

$$\sigma=\tfrac{1}{2}\left\{1-\frac{l}{v}\cdot\frac{dv}{dl}\right\}.$$

An indiarubber tube is used. The lower end is closed by a solid plug; in the upper end a glass tube of small bore is inserted. The distance between the end of the plug and that of the glass tube is taken as l. The indiarubber and glass tubes contain water, and the change in the position of the meniscus when the rubber tube is stretched is observed. From the resulting curve the value of dv/dl is obtained. Proper clamps are provided for securing the rubber tube to the plug and to the glass tube, and other fittings allow these clamps to be held in position on a vertical steel rod. Scales for reading l and the

position of the meniscus are provided. The value of σ is about 0·47. (Searle, "Experimental Elasticity," second edition, 1915.)

4. *Two Methods of Measuring the Surface Tension of Spherical Soap Films.*

In one method the pressure excess due to a curved soap film is measured by aid of what may be called a "viscosity potentiometer." Air from a gasometer or a gas-bag flows through two tubes, AB, BC, in series. The pressure at A is measured by a manometer; the end C is open to the air. From the junction B a side tube leads to a cup with a horizontal circular rim on which a soap film is placed. On account of the viscosity of the air, there is a fall of pressure along each flow tube. For a given flow of air, the fall of pressure in either tube is proportional to the length of the tube and inversely proportional to the fourth power of its internal radius. The excess of the pressure at B over that of the atmosphere causes the film to become part of a sphere. From the distance of the highest point of the film above the plane of the rim and from the radius of the rim, the radius, r, of the film can be computed. If the length of the tube AB be l_1, and its internal radius be a_1, and if l_2 and a_2 be the length and radius of BC, and if the pressure excess at B be p_B and that at A be p_A, then

$$\frac{p_A - p_B}{p_B} = \frac{l_1 a_2^4}{l_2 a_1^4}, \quad \ldots \ldots \quad (1)$$

from which p_B can be found; p_B in the experiment shown corresponded to about 0·03 cm. of water. The surface tension, T, of the film is then found from $4T = rp_B$. It is assumed in (1) that the variation of density of the air along ABC is negligible.

In the second method the pressure excess is measured by a device employed by Mr. J. D. Fry in the calibration of his micro-manometer ("Phil. Mag.," April, 1913, p. 494). The method depends upon the difference of density between cold and hot air at the same pressure. A metal tube, ABCD, has the two parts, AB, CD, each a few centimetres long, at right angles to the main portion BC, which is about 1 metre in length. This bent tube is surrounded by a second tube, which is used as a steam jacket for heating the tube ABCD. The parts AB, CD are horizontal and the tube may, if desired, be rotated about CD as a horizontal axis. The inner tube passes out of the steam jacket through rubber bungs. The opening D is connected by

a horizontal rubber tube to a cup with a cylindrical rim on which a film is formed; the lower end A is open to the atmosphere. If A is at a depth z below D, the pressure excess, p, within the bubble is given by

$$p=gz(\rho_1-\rho_2),$$

where the densities ρ_1 and ρ_2 of the cold and hot air are given by

$$\rho_1=\rho_0 P t_0/p_0 t_1, \quad \rho_2=\rho_0 P t_0/p_0 t_2.$$

Here ρ_0 is the density of air at normal pressure p_0 and normal absolute temperature t_0^0, P is the atmospheric pressure at the time of the experiment, t_1 is the atmospheric temperature near the tube and t_2 is the temperature of the steam. The radius, r, of the bubble is measured and the surface tension is calculated by $T=\frac{1}{4}rp$. ("Proc.," Cambridge Phil. Soc., Vol. XVII., p. 285.)

5. *A Simple Viscometer for Very Viscous Liquids.*

If the space between two coaxial cylinders of radii a, b, and of length h be filled with viscous liquid, the viscosity μ is given in terms of the couple G, which maintains the inner cylinder in motion about its axis with angular velocity ω relative to the outer fixed cylinder, by the equation

$$\mu=\frac{G(a^2-b^2)}{4\pi\omega h a^2 b^2}. \quad \ldots\ldots\ldots \quad (1)$$

The apparatus exhibited to the Society is adapted for finding the viscosity of treacle. The axes of the cylinders are vertical and the outer cylinder is fixed. The inner cylinder is carried by an axle furnished at each end with a pivot working in a fixed bearing. Two threads wound round a drum fixed to the axle pass over ball-bearing pulleys and support weights which drive the inner cylinder round at a slow speed when the space between the cylinders contains treacle. The angular velocity is found to be proportional to the driving couple.

Means are provided for varying the length of the inner cylinder which is exposed to the action of the treacle and in this way a correction is found for the lower end, where the motion of the liquid differs from that contemplated in the theory leading to (1). The arrangement is self-contained so as to avoid the changes of temperature which would result if fresh treacle were added to the system. When the correction has been found, the value of μ can be found from the equation.

At 12°C. the viscosity of treacle is about 400 in C.G.S. units, and that of "golden syrup" is about 1,000; the viscosity of water at the same temperature is 0·0146. ("Proc.," Cambridge Phil. Soc., Vol. XVI., p. 600.)

6. *Experiments with a Prism of Small Angle.*

When a ray in a principal plane passes nearly symmetrically through a prism of small angle i radians placed in air, the deviation, D_1, is nearly independent of the angle of incidence and has the approximate value

$$D_1=(\mu-1)i, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where μ is the refractive index of the prism.

If the prism is placed symmetrically in a tank with parallel glass ends, the deviation is

$$D_2=(\mu-\mu_2)i, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

where μ_2 is the refractive index of the liquid contained in the tank.

The cross-wire of a collimator set for "infinity" is illuminated by sodium light. The direction of the emergent rays corresponding to the cross-wire is observed by a goniometer, the image of the collimator wire being focussed on the cross-wire of the goniometer.

The difference of deviation observed when the prism is turned in air from one position of minimum deviation to the other gives $2D_1$, and the corresponding difference when the prism is immersed in the liquid gives $2D_2$. If a liquid of known index, such as water, is used, (1) and (2) give the angle of the prism and its refractive index. When these have been found, the refractive index μ_3 of any other liquid can be found from the deviation D_3 observed when the tank contains that liquid.

Thus $$D_3=(\mu-\mu_3)i \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

The prisms supplied by opticians for use in spectacles are suitable for this experiment. ("Proc.," Cambridge Phil. Soc., 1915.)

7. *Revolving Table Method of Determining the Curvature of Spherical Surfaces.*

In this method, designed by the author in conjunction with Mr. A. C. W. Aldis and Mr. G. M. B. Dobson, the radius of a

spherical reflecting surface is found directly from two readings on a straight uniformly divided scale, without corrections or calculations of any sort.

A table turning about a vertical axis is required; the plane of the top of the table is normal to the axis of revolution, and the top carries a straight scale, against which slides a carriage bearing the spherical surface. When the apparatus is in adjustment, the straight line described by the centre of curvature of the spherical surface, when the carriage slides along the scale, *intersects* the axis of revolution of the table. The position of the carriage relative to the table top, when the centre of curvature lies on the axis of revolution of the table, will be called the *first position*. It the table be turned through any angle about the vertical axis, when the carriage is in the first position, the only effect of the motion is to substitute one part of the spherical surface for another. Hence, if rays from an object fall upon the surface, the reflected rays will be unaffected by the motion. This furnishes a means of setting the surface in the first position.

The carriage is now moved into a *second position*, in which the vertical axis of the table is a tangent line to the spherical surface. If, now, the table be turned about its axis, a grain of lycopodium placed on the surface at the point of contact of the vertical tangent line will remain stationary.

The radius of curvature of the surface is given by the difference of the two scale readings of the carriage in the first and second positions.

The adjustments of the surface to be tested are facilitated by the use of a small lathe head to form the "carriage." ("Phil. Mag.," February, 1911, pp. 218-224. Also "Proceedings" of the Optical Convention, Vol. II., p. 161, 1912.)

8. *Experiments Illustrating Flare Spots in Photography.*

When light from a point S falls on a simple thin lens of focal length f, most of it passes through the lens, and forms an image of S. But some of the light suffers two reflections within the lens, and this light gives rise to a second image of S of small intensity, the corresponding focal length being $(\mu-1)f/(3\mu-1)$, where μ is the refractive index. This image is called a "flare spot." When two lenses are used there are six flare spot images of any object formed by twice reflected rays, and with t lenses there are $t(2t-1)$ such images.

In the case of two thin lenses AB (radii a, b) and CD (radii

c, d) placed with the faces B, C in contact, the "powers" for the six flare spots are as follows, where the subscript letters indicate the surfaces at which reflection has occurred :—

$$P_{DC}=M+N_2, \qquad P_{DB}=M+N_2+W,$$
$$P_{DA}=M_2+N_2+W, \qquad P_{CB}=M+N+W,$$
$$P_{CA}=M_2+N+W, \qquad P_{BA}=M_2+N.$$

Here M is the "power"=(focal length)$^{-1}$ of AB, and M_2 is the power of AB for the flare spot when AB is used alone, and N and N_2 are the corresponding powers of CD. Also $W=-2(1/b+1/c)$. Each radius is counted positive when the corresponding surface is convex. Following the rule of the practical opticians, the focal length of a converging lens is counted positive.

If each of the two lenses is a converging meniscus, M, M_2, N, N_2 are positive, and W is also positive if the concave surfaces face each other. In this case the system has a positive power for each of the six flare spots, and thus six real flare spot images of a real object can be formed. "Periscopic" spectacle lenses are convenient for the experiment. The values of the five quantities M, M_2, N, N_2, W are found by experiment, and the six powers P_{DC}, &c., are calculated from them. The values are found to be in good agreement with the six powers found when the complete system of two lenses is used.

If a suitable converging lens is placed between the two meniscus lenses, the 15 secondary images are easily seen. ("Proc.," Cambridge Phil. Soc., Vol. XVII., p. 205.)

9. *Determination of the Effective Aperture of a Photographic Lens.**

The stops of a lens are marked with such symbols as $f/8$. The symbol $f/8$ means that the effective diameter of the stop is one-eighth of the focal length of the lens system. The effective diameter of the stop is not the diameter of the hole in the diaphragm, but is the diameter of that incident beam of rays parallel to the axis which in its passage through the system exactly fills the opening in the actual diaphragm.

When the stop is placed in front of the lens system, its effective diameter is simply equal the actual diameter of the stop itself.

In most cases the stop S is placed between the components of

the lens system. Let T be the image of S formed by the front lens L. Then S is also the image of T, and a ray, RQ, which before incidence on L is parallel to the axis, and is directed to a point, Q, on the edge of T, will, after passing through L, pass through P, the corresponding point on the edge of S. Hence, the distance of RQ from the axis, and, therefore, the radius of T, is equal to the effective radius of the stop.

The effective diameter may be measured by a microscope mounted on a sliding carriage. The axis of the microscope is parallel to the axis of the lens system, and the carriage moves at right angles to that axis. The microscope is focussed through the lens L upon the stop, and the diameter of the image T is measured by aid of the sliding carriage.

If a point source of light is placed at that principal focus, F, of the system which lies on the photographic plate when distant objects are in focus, the rays from this source will, after passing through the system, form a parallel beam. If this beam is received on a translucent screen, A, placed in front of L, the diameter of the luminous patch is the effective diameter of the stop.

Since an infinitely small source cannot be obtained, a correction becomes necessary.

In practice, a small hole of diameter h in a metal plate in the focal plane is illuminated by a flame to serve as the source of light. If the diameter of the bright patch on the translucent screen is c, then the effective diameter a of the stop is given by

$$a=c-xh/f,$$

where f is the focal length of the complete lens system, and x is the distance of the image T from the matt surface of the screen. The distance x is measured by aid of the microscope and sliding carriage, the axis of the microscope being now parallel to the direction of motion of the carriage.

The determination of x may be avoided if two holes of diameter h_1, h_2 are used, the corresponding diameters of the bright patch being c_1, c_2. Then

$$\frac{c_1-a}{c_2-a}=\frac{h_1}{h_2},$$

whence

$$a=c_1-\frac{h_1(c_1-c_2)}{h_1-h_2}.$$

(" Proc.," Cambridge Phil. Soc., 1915.)

10. *The Comparison of Nearly Equal Resistances.*

Four resistance coils A, B, C, D, are arranged to form the four sides of a Wheatstone's quadrilateral. The coils C, D are approximately equal, but, as their ratio is eliminated, it is not necessary to know it. A balance is obtained by shunting A, B with large resistances a_1, b_1*. The coils A and B are then interchanged, and a fresh balance is obtained by shunting them with a_2 and b_2. Then

$$C\left(\frac{1}{A}+\frac{1}{a_1}\right)=D\left(\frac{1}{B}+\frac{1}{b_1}\right)$$

and

$$D\left(\frac{1}{A}+\frac{1}{a_2}\right)=C\left(\frac{1}{B}+\frac{1}{b_2}\right).$$

Hence

$$\left\{\left(\frac{1}{A}+\frac{1}{a_1}\right)\left(\frac{1}{A}+\frac{1}{a_2}\right)\right\}^{\frac{1}{2}}=\left\{\left(\frac{1}{B}+\frac{1}{b_1}\right)\left(\frac{1}{B}+\frac{1}{b_2}\right)\right\}^{\frac{1}{2}}.$$

In practice, A is so nearly equal to B and C is so nearly equal to D that a_1, a_2, b_1, b_2 are very large compared with A and B, and the arithmetic means may be used instead of the above geometrical means. Then

$$\frac{1}{A}+\tfrac{1}{2}\left(\frac{1}{a_1}+\frac{1}{a_2}\right)=\frac{1}{B}+\tfrac{1}{2}\left(\frac{1}{b_1}+\frac{1}{b_2}\right),$$

which gives B in terms of A when the shunts are known. The details of a complete intercomparison of four coils are given. ("Proc.," Cambridge Phil. Soc., Vol. XVII., p. 340.)

ABSTRACT.

The following experiments were shown :—

1. Harmonic motion of a rigid body suspended by a torsion wire.
2. Determination of gravity by a rigid pendulum.
3. Determination of Poisson's ratio for indiarubber.
4. Two methods of measuring the surface tension of spherical soap films.
5. A simple viscometer for very viscous liquids.
6. Experiments with a prism of small angle.
7. Revolving table method of determining the curvature of spherical surfaces.
8. Experiments illustrating flare spots in photography.
9. Determination of the effective aperture of a photographic lens.
10. The comparison of nearly equal electrical resistances.

* It is not necessary, as a rule, to shunt both A and B, but it is simpler to suppose them both shunted in deriving the formula. If B is not shunted the value of b is infinite.

DISCUSSION.

Mr. DUDDELL complimented Dr. Searle on the simplicity and ingenuity of the arrangements. Some apparatus of the older style was so complicated that students took longer to understand the apparatus than to understand the principles it was intended to illustrate.

Principal SKINNER mentioned that he had once determined Poisson's ratio for cork in a simple manner with an old champagne cork. From the markings of the cork the length and diameter of the compressed portion could be found. The cork was then boiled to restore it to its original shape, and from its change in dimensions one could determine Poisson's ratio roughly.

Mr. F. E. SMITH associated himself with Dr. Searle's remark that small corrections constituted the fun of physics, in addition to being, very often, of fundamental importance. He hoped that when Dr. Searle had finished his other press work he would turn his attention to a book on practical physics.

Mr. S. D. CHALMERS thought the experiment shown with the small prism brought out very clearly the connection between a prism and a lens. In practice, when measuring the aperture of a photographic lens it was not convenient to use a large hole as done by Dr. Searle, as the mount cut off some of the sloping rays, and one had to do the best possible with a small hole.

Dr. RUSSELL said that personally he was much indebted to Dr. Searle, who had given him valuable assistance many times in the past.

Dr. SEARLE thanked the various speakers for their remarks.

IX. *The Vacuum Guard Ring and its Application to the Determination of the Thermal Conductivity of Mercury.* *By* H. Redmayne Nettleton, *B.Sc., Assistant Lecturer in Physics at Birkbeck College.*

1. *Introduction.*

During a research recently described before this society on the thermal conductivity of mercury by a method of impressed velocity, the author had occasion to employ a wide vacuum-jacketed tube. ("Proc." Phys. Soc., London, Vol. XXVI., December, 1913.) Though the tube was unsilvered and unprotected by any non-conductor of heat, the reduction of emissivity loss was so marked that calculation showed the possibility, in spite of the relatively small value of the conductivity of mercury, of obtaining, within a similar vessel in which the mercury was at rest, a temperature gradient sufficiently constant over a range of several centimetres to enable a direct determination of the thermal conductivity to be made.

The chief disadvantage of the ordinary guard ring lies in the uncertainty of the area from which the heat is collected, an uncertainty which becomes more pronounced if means other than the ice calorimeter be employed to measure the quantity of heat transmitted. The efficient substitution of a vacuum for a guard ring would remove the objection and open up the way for further development of both electrical and continuous-flow methods of calorimetry.

The measurement, by the steady flow of water, of the quantity of heat transmitted by conduction was first carried out by Callendar and Nicolson ("Proc." Inst.C.E., 1898) in their experiments on a cast-iron bar. Consistent results were obtained, and the calorimetry was very satisfactory, but the temperature gradient down the bar being far from linear its value near the entrance to the calorimeter, where some distortion of the isothermals is likely, had to be deduced by interpolation. Again, Searle ("Phil. Mag.," Jan., 1905, p. 125) has used the continuous flow method in his elegant laboratory apparatus for the determination of the thermal conductivity of copper. The bar being some 4 cm. in diameter and well protected by non-conducting material, the assumption is made that the temperature gradient is sufficiently linear to allow a measurement of it to be made from two thermometers 7·5 cm. apart. The thermal conductivity of mercury, however, is

only about one-fiftieth that of copper. None the less it is shown below that by virtue of the vacuum surrounded with cotton wool the heat lost in a length of some 12 cm. must be a negligible fraction of the quantity of heat transmitted. The measurement of temperature gradient can thus be very accurately effected with the aid of a single thermo-junction carried by a cathetometer, and at the same time the horizontal nature of the isothermals can be readily demonstrated, while the quantity of heat transmitted by conduction can be measured by a modification of Searle's method.

The determination of the thermal conductivity of mercury by a method so direct and simple when once the vacuum vessel has been constructed would seem warranted by the great difference in the values obtained by different experimenters for a substance for which the other constants are known to a particularly high accuracy.

It is hoped that the usefulness of the vacuum guard ring will not be limited to the case of mercury, for it is possible that it may afford useful information with respect to the relative conductivities of other liquids; more particularly experiment indicates that it is likely to prove useful for determining the thermal conductivity of metallic solids.

2. *Description of Apparatus and Method of Experimenting.*

(*a*) *The Vacuum Vessel.*—The specially constructed vacuum vessel is seen at V in Fig. 1. The inner tube is of uniform cross-section about 4·9 cm. in diameter and 40 cm. long. The outer tube, about 20 cm. long, is sealed at both its extremities to the inner one. Thus, there is no free end, as in an ordinary vacuum vessel, to permit of differences of contraction on cooling between two portions, but some elasticity is afforded by making the outer vessel concertina-shaped or corrugated. Owing to this double sealing, combined with the large width of tubing, some difficulty was experienced in obtaining such a vessel, which was at length supplied to the author from Germany early in 1913 by Messrs. Müller, Orme & Co. The vessel is silvered, and was exhausted to a high standard on a Gaede molecular pump.

The part of the inner tube not vacuum-jacketed is fitted with a long and annular heater, A. The lower part is fitted with a wide rubber cork, R, which carries the heat collecting spiral of copper S wound round the turned-down portion of the brass disc D. This disc, which was supported on the rubber

cork by a piece of ebonite let into both, was of a diameter slightly less than that of the inner tube of the vacuum vessel, and was filed in one place to permit the thermo-electric carrier tube T to pass it; it serves to collect the heat with a minimum

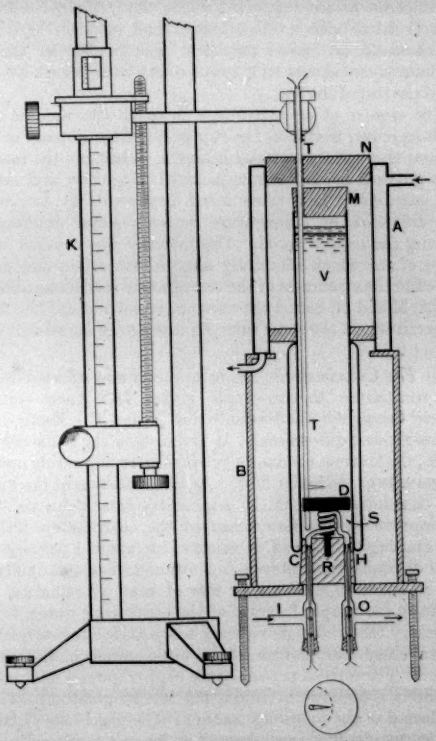


FIG. 1.

of convection, and at the same time provides a horizontal surface approximately isothermal. Both the brass disc and copper spiral were insulated with enamel to protect them from

the action of the mercury. The cork R was well secured by letting first seccotine and subsequently club enamel down on to the edges of it by means of a glass tube passing through the filed cavity in the brass disc. The vacuum vessel thus adapted for experiment was supported within the specially constructed box B fitted with levelling screws and made in two halves which could be locked together. The portion of the tube vacuum-jacketed was well wrapt round with cotton wool and filled the rest of the box.

The essence of an experiment is to fill the vacuum vessel with mercury, heat it at the top by circulating steam or other vapour through the annular heater A, determine the temperature gradient in the vacuum-jacketed portion *vv*, and measure the quantity of heat transmitted downwards by determining the difference of temperature between water entering and leaving the copper spiral. The mercury was covered with a layer of tar which effectively damped vibration, and greatly added to the steadiness of the temperature conditions obtained. Corks, M and N, closed the vacuum vessel and annular heater respectively, at the same time permitting the movement of the carrier tube T.

(*b*) *The Calorimetry.*—The calorimetry was effected with an iron-constantan thermo-couple giving 44·2 micro-volts per degree Centigrade, and situated at H and C. Early experiments showed differences of 3½ per cent. in the value obtained for K, the thermal conductivity of mercury depending upon the magnitude of the water flow. As the limitation of the method was thus found to lie in the calorimetry great pains was taken to improve it. As now arranged the calorimetry, although still limiting the method, is consistent to about 1 per cent. The chief difficulty encountered in continuous flow calorimetry lies not so much in diminishing loss of heat by radiation as in ensuring the proper mixture of the outflowing water, for it is necessary that the higher steady temperature registered is the true average temperature over a cross-section of the warmed water. The vertical portion of the copper spiral is undoubtedly helpful in attaining this object, but further great improvement is effected in the following manner: The glass tube O holding the warm junction was plugged at its upper extremity with copper foil through four or five narrow channels in which the water was compelled to flow; close underneath the common outflow was the warm junction insulated and subsequently tipped at its head with copper foil, the head thus closely fitting

the glass tube. This arrangement proved very satisfactory, and probably the limit of accuracy in the calorimetry is determined by the conduction of heat across the rubber cork from the hot to the cold stream. The flow of water which was air-free by exhaustion was controlled by a constant head of sufficient height to force it through the fine channels in the copper foil, and through a greater resistance consisting of a drawn out capillary of glass, the height of which could be varied. The

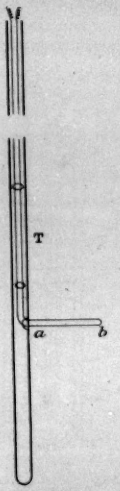


FIG. 2.

constancy of temperature of the inflowing water was much improved by passing it through a "condenser" consisting of a copper tube surrounded by a wider glass tube which extended well up the inlet tube I. (Fig. 1), and conveyed ordinary tap water, which in time became very steady.

(c) *The Measurement of Temperature Gradient and the Cold Junction Thermostat.*—The measurement of the temperature gradient within the portion of the mercury protected by the vacuum was effected by a single insulated iron-constantan thermo-junction. The thin insulated wires were contained

within the straightened vertical glass carrier tube T (Figs. 1 and 2), of less than 6 mm. diameter, and were fused through the glass at *a* (Fig. 2) with inappreciable external distortion, the wires from *a* to the junction *b* being close together, and in the same horizontal plane. The length *ab* was such that when the tube T was resting against the inner wall of the vacuum vessel the point *b* would just reach the centre of the latter. The thermo-electric carrier tube T was carried by the cathetometer K (Fig. 1), specially fitted with complete translational and rotational adjustments. The thermo-junction could thus be raised or lowered vertically through any measurable distance

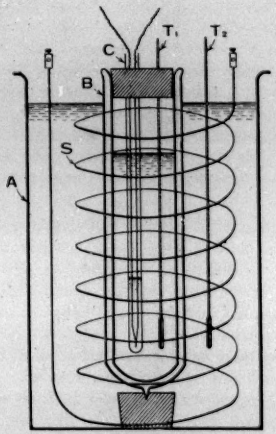


FIG. 3.

or rotated over a horizontal cross-section till the point *b* touched the walls of the vacuum vessel.

It will be seen that it is often necessary to maintain the other or "cold junction" of this circuit used for measuring the gradient at a steady temperature near the lower limit of the range measured by the hot one. The thermo-electric "thermostat" seen in Fig. 3 proved very efficient, the temperature certainly remaining constant to within 0·01°C. over the time of each independent experiment. A is a large outer vessel surrounded with cotton wool, and containing the

Dewar cylindrical vacuum vessel B, supported on a cork as shown. The thermo-junction is enclosed within the small glass tube C containing paraffin oil, and passing through the cork D. A and B are filled with water to the levels shown at a temperature approximate to that required, the water in B being covered with a layer of Fleuss pump oil. T_1 and T_2 are thermometers reading to 0·1°C. S is a coil of thick constantan wire lying between B and the walls of A. By passing a suitable current through S the temperature of the thermometer T_2 can be kept approximately constant at the desired value. When steady T_2 is observed at intervals of one hour or so, and the current, if necessary, is raised or lowered by means of a step rheostat. By this means, the mass of water being large, T_2 usually keeps constant to 0·1°C., showing that though the temperature may vary somewhat in different parts of A the temperature at any one place changes very slowly. The small temperature oscillations in A, however, are so damped by the vacuum that change of temperature in T_1 can often not be detected for hours. T_1 is uaually about half a degree below T_2, and very uniform throughout its mass. This arrangement is not only useful for holding a "cold junction," but may be used as a constant temperature source when calibrating thermo-couples in micro-volts per degree or when maintaining sodium sulphate at its transition point. With practice it can be very readily adjusted.

(*d*) *Method of Comparing the Two Fundamental Temperature Differences.*—Experiment shows that the temperature gradient within the portion of the mercury which is vacuum-jacketed may be regarded as constant. Let A be the effective cross-section of the vacuum vessel, and φ the rise of temperature in a space of 1 cm. Let θ be the difference of temperature on the same scale between the inflowing and outflowing water, the rate of flow being m grams per second. Then, taking the specific heat of water as unity we have for K the thermal conductivity of mercury :

$$KA\varphi = m\theta,$$

or,

$$K = m/A \times \theta/\varphi.$$

The ratio of two temperature intervals is thus required.

Now, as the relation between temperature and E.M.F. is very strictly linear for the iron constantan couple over the operative range 15°C. to 45°C., and as over this range the micro-volts per degree for both the calorimetric and gradient thermo-couples were found to agree to 1 part in 650, it is clear that it is only

necessary to compare the E.M.F. given by the two sets of couples. For this purpose both deflection and potentiometer methods were employed each having its special advantage.

While employing the deflection method the resistances of the two thermo-couples were adjusted to equality by suitably choosing the position of join of the two constantan wires of the calorimetric couple. This could be easily effected to an accuracy of 0·01 ohm and tested at any moment, and as an additional 30 to 50 ohms was always in circuit with the couples and galvanometer, change of resistance with rise of temperature on moving the gradient thermo-junction was quite negligible. The deflection produced by each circuit on inserting the proper

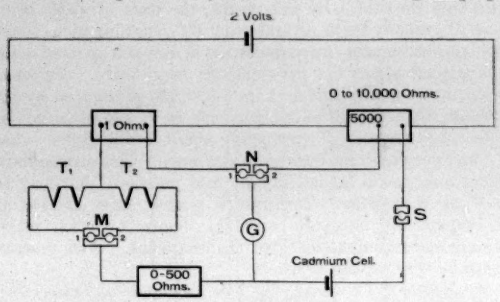


FIG. 4.

plug key was then proportional to the temperature interval. This was carefully verified directly by testing both couples simultaneously for one and the same temperature difference.

While the deflection method was very rapid and sufficiently accurate for experiments on the conductivity of mercury, yet, owing to the necessary limitation of the galvanometer scale, it was found inferior to the potentiometer method when attempting to test the exact nature of the temperature gradient or when measuring temperature differences directly in degrees Centigrade. In the absence of a potentiometer suitable for thermo-electric work, the arrangement shown in Fig. 4 was found very satisfactory. Briefly, the E.M.F. of either thermo-couple could be balanced against that across a standard ohm in series with an accumulator and a standard box of 0 to 10,000 ohms fitted with a dividing plug. The reciprocal of

the total resistance needed for a balance is, therefore, proportional to the temperature interval being measured as long as the total external voltage remains constant. This voltage can be measured in a moment by dividing the standard box into two portions, one of 5,000 ohms and the other slightly less, and such that the E.M.F. across it exactly balances that of a standard cell. The micro-volts given by either couple, and hence the interval in degrees Centigrade, can thus be easily calculated if required. A slight simplification of the connections reduces them to those necessary for the deflection method. Typical results illustrative of both methods are given in the next section. Precisely the same arrangement served for measuring in degrees Centigrade the difference of temperature between the inflowing and outflowing water in the experiments on the electrical supply of heat to the vacuum vessel, which are described in section 4.

Every care was taken to minimise local thermoelectric effects. All four joins of iron to copper wire were in small tubes side by side contained within a large test tube holding paraffin oil, and immersed in a Newton's annular cooling calorimeter filled with water, an arrangement which in previous work had proved satisfactory. Brass plug keys for joining copper wires were tested, and found suitable both from the thermoelectric and resistance standpoints. The keys should be large, and the plug holes well away from the copper joins. Both couples registered satisfactory zeros when the hot and cold junctions were placed side by side in the same stirred water, while the slight disturbance which was caused by deliberately touching the keys with the hand at the join of the two metals quickly subsided on removing the fingers. Repeatedly removing and inserting the actual plugs caused no appreciable effect. Joins of copper, brass and manganin can easily be rendered thermo-electrically safe, but with other metals special care has to be exercised.

3. *Results of Experiments.*

(*a*) *Horizontal Nature of Isothermals in Mercury.*—It will be seen from figures given in sub-section (*b*) below that a temperature gradient of just over 3°C. per centimetre was obtained over a length of some 12 cm. when the vacuum vessel filled with mercury was heated at the top by steam. On rotating the thermo-junction over a horizontal section within this range there was no appreciable alteration of galvanometer deflection even when the junction was on the point of touching the walls,

and the protruding wires were lying along the side of an inscribed hexagon. The only exception to this is to be found in the 2 cm. immediately above the disc D at the lower "end" of the column. This is illustrated by the figures in the next column, which show that the distortion is of an unsymmetrical character, and accounts for the slightly greater value of the temperature gradient seen in Tables III. and IV. of subsection (*b*) in the same region.

TABLE I.—*Showing Nature of Isothermals.*

Cathetometer reading.	Relative Temperatures with thermo-junction lying at the positions:—		
	Extreme left.	Middle.	Extreme right.
65	22·20	21·78	21·50
66	9·30	9·05	8·95
67	−3·60	−3·57	−3·55
68	−16·20	−16·20	−16·16

Thus, in the short space of 2 cm. the horizontal surfaces have become appreciably isothermal. A perhaps equally rigorous proof of this is afforded by the fact that repeated measurements of temperature gradient made over ranges of 7 cm. or 8 cm. in the middle of the vessel never differed by so much as 1 part in 500, whether the thermo-junction was moved vertically through the middle of the column or when almost touching the walls.

(*b*) *The Linear Nature of the Temperature Gradient in Mercury.* The first experiment performed with the apparatus at once revealed the linear nature of the temperature gradient. Boiling alcohol was used to supply the heating vapour, and no tar was covering the mercury surface. Deflections on the galvanometer scale are proportional to temperatures, the actual temperatures at the cathetometer positions marked 66 and 77 being approximately 23·4°C. and 46·2°C. respectively.

TABLE II.—*Showing Relative Temperatures at Definite Cathetometer Positions by the Deflection Method.*

Cathetometer reading.	Deflection.	Cathetometer reading.	Deflection.	Differences per 6 cm.
65	23·35	71	8·25	15·10
66	20·88	72	5·68	15·20
67	18·42	73	3·08	15·34
68	15·88	74	0·68	15·20
69	13·35	75	−1·93	15·28
70	10·75	76	−4·45	15·20
71	8·25	77	−7·00	15·25
72	5·68	78	−9·25	14·93

This result obtained by the deflection method was more than confirmed when the potentiometer method illustrated by the connections in Fig. 4 was employed. Table III. gives the figures for an actual experiment, the reciprocal of the total ohms needed for a balance being proportional to the difference of temperature between the hot and cold thermo-junctions. The surface of the mercury was covered with tar and steam was used as the heating vapour. The cold junction in the thermostat was at 25·43°C., the accumulator was balanced for the Weston cell at 4,848 ohms (viz., 4,846 ohms before the first reading and 4,850 ohms after the last one), and hence at any position of the hot junction where R is the total number of ohms needed for a balance the actual temperature in degrees Centigrade is given by :—

$$\theta = 25{\cdot}43 + W \times \frac{10^6}{44{\cdot}2} \times \frac{9{,}849}{4{,}848} \times \frac{1}{R},$$

where W is the E.M.F. of the Weston cell. Thus, the actual temperatures at cathetometer positions 67 and 76 are 36·61°C. and 64·69°C. respectively which give for the mean value of temperature gradient a value of 3·12°C. per centimetre. In the table below, however, it suffices to give actual readings and relative temperatures :—

TABLE III.—*Showing Relative Temperatures at Definite Cathetometer Positions by the Potentiometer Method.*

Cathetometer position.	Total ohms for balance.	Reciprocal of total ohms $\times 10^7$	Differences.	Differences per 6 cm.
66	5,824	1,717·0	...	...
67	4,185	2,389·5	672·5	...
68	3.270	3,058·1	668·6	...
69	2,684	3,725·8	667·7	...
70	2,277	4,391·7	665·9	...
71	1,977	5,058·2	666·5	...
72	1,748	5,720·8	662·6	3,331
73	1,565	6,389·8	669·0	3,332
74	1,417·0	7,057·2	667·4	3,331
75	1,295·0	7,722·0	664·8	3,330
76	1,192·2	8,387·9	665·9	3,329

The closeness of agreement between the figures in the last column is in part accidental, as the decimal places for the total ohms in the cathetometer positions 74 to 76 were obtained by proportional deflections, and for the other positions the balance

was only found to the nearest ohm. Apart from this each balance was very definite, and there is no doubt that the cathetometer can be set to an exact whole centimetre division with a greater accuracy than fractions can be read by the vernier. A repetition made two days later again gave remarkable closeness. The actual figures of this experiment are tabulated below :—

TABLE IV.—*Relative Temperatures at Cathetometer Positions.*

Cathetometer position.	Total ohms for balance.	Reciprocal of total ohms $\times 10^7$	Differences.	Differences per 3 cm.
66	7,301	1,389·7	...	...
67	4,901	2,040·4	670·7	...
68	3,693	2,707·8	667·4	...
69	2,961	3,377·2	669·4	2,008
70	2,473	4,043·7	666·5	2,003
71	2,122·0	4,712·5	668·8	2,005
72	1,858·6	5,380·4	667·9	2,003
73	Omitted not	observed.	...	...
74	1,488·1	6,720·0	669·8	2,008

The linear character of the temperature gradient was also seen in the direct experiments on the thermal conductivity of mercury described in the next sub-section. It would seem that by moving the thermo-junction, say, from position 67 to position 76 the temperature gradient can be measured to an accuracy ap'proaching 1 part in 1,000.

(c) *Determination of the Thermal Conductivity of Mercury.*—A typical experiment using the deflection method is given below, the order being as follows : At an exact minute a weighed flask was placed in position to receive the outflowing water. The deflection in the calorimetric circuit corresponding to the difference of temperature between the inflowing and outflowing water was then observed. The temperature gradient was next measured by raising the thermo-junction, readings being usually taken at every half-centimetre between the cathetometer positions 67 and 70·5. The calorimetric deflection was again observed, and the flask removed at the next available whole minute. The temperature of the room was then recorded as well as the approximate temperatures of the hot and cold water, as read by thermometers lying in the inlet and outlet tubes I and O of Fig. 1. All readings could be made in an interval of 15 minutes.

TABLE V.—*Typical Deflection Experiment on the Thermal Conductivity of Mercury.*

α Measurement of temperature gradient.

Cathetometer reading.	Deflection.	Cathetometer reading.	Deflection.	Difference.
67·0	15·02	69·0	— 3·75	18·77
67·5	10·26	69·5	— 8·34	18·60
68·0	5·54	70·0	—13·12	18·66
68·5	0·90	70·5	—17·87	18·77

Mean difference, 18·70.

ϕ = Difference per centimetre, 9·35

β other measurements.

Initial calorimetric deflection	21·05
Final calorimetric deflection	21·12
Mean value of θ	21·09
Temperature of inflowing water	17·1°C.
Temperature of outflowing water	24·1°C.
Temperature of room	20·3°C.
Temperature of " cold junction "	35·68°C.
Area of cross-section of mercury	18·47 sq. cm.
Magnitude of flow of water	$0{\cdot}160_5$ gms. per sec.

Whence $$K = \frac{0{\cdot}1605 \cdot 21{\cdot}09}{18{\cdot}47 \times 9{\cdot}35} = 0{\cdot}0196_0 \text{ C.G.S. units.}$$

A typical potentiometer experiment is illustrated by the figures of Table VI. :—

TABLE VI.—*Typical Potentiometer Experiment.*

Initial calorimetric balance	10,670 ohms.
Gradient balance at position 71	2,103 ohms.
Gradient balance at position 67	4,406 ohms.
Final calorimetric balance	10,700 ohms.
Weston cell balance	4,853 ohms.
Magnitude of water flow	0·2400 gms. per sec.
Gradient cold junction	25·70°C.
Temperature of room	20·4°C.
Temperature of outflowing water	22·2°C. (approx.).
Temperature of inflowing water	17·6°C. (approx.).

Whence ϕ the temperature gradient $= \frac{1}{4}\left(\frac{1}{2103} - \frac{1}{4406}\right)$, and θ the calorimetric difference $= \frac{1}{10685}$, giving $K = 0{\cdot}0195_7$ C.G.S. units.

The mean value obtained for K the thermal conductivity of mercury as the result of 24 experiments under varying flows is $0{\cdot}0196_0$ C.G.S. units, the range of temperature being 35°C. to 45°C. No results have been rejected in this sequence. The highest value obtained was $0{\cdot}0198_2$ and the lowest $0{\cdot}0194_2$, but 19 of the 24 values do not differ by as much as 0·5 per cent.

from the mean. The greatest flow of water employed was 19·14 grams per minute, and the slowest flow 9·5 grams per minute. Tables VII. and VIII. below summarise the results of these experiments :—

TABLE VII.—*Values of* K *by Deflection Experiments.*

m=flow of water in grms. per second.	θ=deflection of calorimetric circuit.	ϕ=mean change of deflection per cm. of gradient circuit.	dt=approx. difference of temp. between inflowing and outflowing water in deg. C.	K
$0{\cdot}223_7$	16·16	10·08	5·5	$0{\cdot}0194_3$
$0{\cdot}275_3$	13·28	10·06	4·7	$0{\cdot}0196_0$
$0{\cdot}160_5$	21·09	9·35	7·0	$0{\cdot}0196_6$
$0{\cdot}222_1$	16·31	$9{\cdot}92_5$	5·6	$0{\cdot}0197_6$
$0{\cdot}275_7$	13·48	10·15	4·7	$0{\cdot}0198_5$
$0{\cdot}314_7$	15·14	$13{\cdot}03_5$	4·1	$0{\cdot}0197_5$
$0{\cdot}255_5$	18·28	12·84	4·9	$0{\cdot}0196_5$
$0{\cdot}212_9$	22·24	13·08	5·5	$0{\cdot}0196_0$
$0{\cdot}210_4$	22·38	$13{\cdot}02_5$	5·5	$0{\cdot}0195_7$
$0{\cdot}177_2$	26·19	12·88	6·4	$0{\cdot}0195_1$
$0{\cdot}260_0$	18·64	$13{\cdot}32_5$	4·7	$0{\cdot}0196_9$
$0{\cdot}237_1$	20·18	13·25	5·1	$0{\cdot}0195_5$
$0{\cdot}206_0$	22·99	13·10	5·8	$0{\cdot}0195_7$

Mean value of K=$0{\cdot}0196_4$.

TABLE VIII.—*Values of* K *by Potentiometer Experiments.*

m=flow of water in grms. per second.	θ=mean recip. ohms of cal. circuit $\times 10^7$.	ϕ=change of recip. ohms per cm. of gradient circuit $\times 10^7$.	dt=approx. difference of temp. of hot and cold water in deg. C.	K
$0{\cdot}240_0$	935·6	621·2	4·6	$0{\cdot}0195_7$
$0{\cdot}232_2$	1,026	660·2	5·1	$0{\cdot}0195_4$
$0{\cdot}247_2$	986·9	677·4	4·8	$0{\cdot}0195_0$
$0{\cdot}319_1$	792·1	696·2	4·0	$0{\cdot}0196_6$
$0{\cdot}183_2$	1,294	658·6	6·2	$0{\cdot}0195_0$
$0{\cdot}217_2$	1,117	667·8	5·5	$0{\cdot}0196_8$
$0{\cdot}214_5$	1,126	669·4	5·5	$0{\cdot}0195_4$
$0{\cdot}255_0$	939·8	663·0	4·4	$0{\cdot}0195_7$
$0{\cdot}221_0$	1,070	658·8	5·0	$0{\cdot}0194_3$
$0{\cdot}168_5$	1,370	639·2	6·5	$0{\cdot}0195_9$
$0{\cdot}158_5$	1,451	636·2	6·7	$0{\cdot}0195_5$

Mean value of K=$0{\cdot}0195_6$.

(*d*) *Experiments on Temperature Gradient and Isothermals in Water.*—Numerous experiments were performed on the temperature gradient and the isothermals in water, but it is perhaps out of place in this Paper to go into detail. Suffice it here to say that the gradient experiments confirm the efficacy of the vacuum guard ring. With respect to the isothermals in water,

they, too, are approximately horizontal over the cathetometer range 76 to 70, where very satisfactory measurements of temperature gradient can be made. But distortion of a non-symmetrical character is very pronounced in the neighbourhood of the collecting disc D (Fig. 1), and persists, though in gradually diminishing degree, till the thermo-junction has been raised four or five centimetres.

(*e*) *Determination of Area of Cross-section of the Vacuum Vessel.*—The mean area of cross-section of the operative part of the vacuum vessel could easily be found to an accuracy of 1 part in 1,000, with the aid of mercury and an electrical point contact carried by the cathetometer used previously for measuring the gradient. The vessel is remarkably uniform over the 12 cm. previously traversed by the thermo-junction, the mean area of cross-section over any particular length of 5 cm. taken not differing from that over any other similar length by as much as 1 part in 500. The mean area of cross-section at 20°C. is 18·74 sq. cm. The cross-section of the thermo-junction carrier tube was 0·27 sq. cm., and hence the effective mercury area is 18·47 sq. cm.

4. *Electrical Supply of Heat to the Vacuum Vessel.*

During the tedious process of improving the calorimetry it was very useful to be able to supply to the vacuum vessel an amount of heat generated electrically at a rate approximately equal to that transmitted down the vessel when full of mercury and heated at the top by steam. By this means the constancy of the heat collected under different rates of water flow could rapidly be tested. Further, the better the design of the heater, from the point of view of preventing heat loss from the top of the vacuum vessel, the more valuable does it become as a check on the reliability of the continuous flow calorimetry.

The heater (Fig. 5) consists of slightly less than 2 metres of thin manganin wire, gauge 34, wound as nearly as possible in a horizontal plane over fine saw cuts in an annular fibre frame, F, of about 4·2 cm. diameter. This frame is supported by a bridge and rod of fibre, the latter being let in to the cork cylinder C which was turned down till of a diameter such as to fit the vacuum vessel very closely. The ends of the manganin heating coil were soldered to thin copper wires of gauge 32, which on leaving the frame were brought up along slits in the cork for about 10 cm., where they were again joined to thicker copper wire of gauge 22, which led to the double terminals

T, T on the fibre handle. The cork cylinder C reached beyond the top of the vacuum vessel when the frame F was just touching the brass disc. The heating coil and leads, which were of about 16 ohms resistance, were insulated with two or three applications of club enamel, and dried at a high temperature by the electric current; they were then varnished with

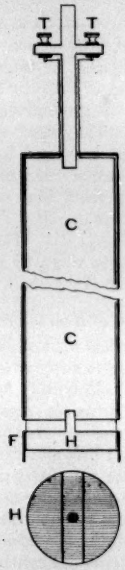


Fig. 5.

velure and again dried. Thus treated the heating coil could safely be immersed in mercury.

When using the heater a little mercury was poured into the vacuum vessel till it covered the horizontal collecting disc (Fig. 1) to a depth of about 5 mm. The heater was then pushed down the vacuum vessel till the bottom of the fibre-ring lay on this disc, and displaced sufficient mercury to completely cover the insulated coil. A constant current derived

from accumulators of 0·55 ampere as registered on a 0·0—0·6 Paul ammeter was then passed through the heater, while the heat was collected and measured by flowing water, the same arrangement as in Fig. 4, serving to measure the difference of temperature.

The heat generated by the electric current between the terminals of the heater was measured to an accuracy of 0·5 per cent., use being made of the copper voltameter, taking the proper precautions, to measure the value of the current, which would thus be correct to about 1 part in 500. The resistance of the heater and external leads were measured on a Post Office box on which they could be switched by the same key that broke the main current. The resistance of the heater between its terminals was found to be 15·93 ohms when cold and 15·95 ohms when tested isntantly after breaking the heating current.

It will be realised that the copper wires soldered on to the manganin coil were so chosen in gauge that while the heat generated in them electrically was small the heat conducted away by them was also inappreciable. It is easy to show the heat lost by conduction in an infinite wire of perimeter p and area of cross-section a in a uniform enclosure and dipping into a constant temperature source at U degrees above the enclosure is, when conveying electric current, less than a quantity $H=U\sqrt{EpKa}$ calories per second, where E is the coefficient of emissivity loss and K the thermal conductivity of the wire. Moreover, U, the temperature of the mercury layer touching the cork, can be roughly estimated from a knowledge of the distance between the heating coil and the collecting disc and the value of the temperature gradient, known from the conductivity experiments, which will permit the passage downwards of the heat generated and collected. In this way it is estimated that the heat lost per second by the two wire leads of gauge 32 is of the order 0·005 calorie per second, which is almost as great as that lost by the entire cork protected by the vacuum jacket. The resistance of the leads within the terminals, but outside the mercury was 0·07 ohm, so that the rate of generation of heat within them is again 0·005 calorie per second. It would seem likely, then, that there is a loss of quite 1 per cent. of the heat generated by the electric current between the terminals of the heater.

The following are some of the results obtained: A constant current of $0{\cdot}546_3$ ampere was passed through the heater of resistance 15·95 ohms between the terminals. The rate of

generation of heat, taking J at 20°C. as 4·180 joules, is thus 1·139 calories per second. The heat collected is seen from the following table :—

TABLE IX.—*Showing Heat Collected by the Calorimeter from an Electrical Heater.*

m=rate of flow of water in grms. per second.	$d\theta$=temp. difference between hot and cold water.	$m \times d\theta$=quantity of heat collected per second.
$0{\cdot}301_0$	$3{\cdot}70_9$	1·116
$0{\cdot}261_0$	$4{\cdot}23_2$	1·105
$0{\cdot}243_7$	$4{\cdot}55_2$	1·109
$0{\cdot}242_0$	$4{\cdot}60_7$	1·115
$0{\cdot}204_7$	$5{\cdot}41_2$	1·108
$0{\cdot}162_1$	$6{\cdot}83_2$	1·107

The mean of 12 consecutive experiments gives 1·110 calories per second as the rate at which heat is collected, which thus falls short of the heat generated electrically by just over 2·5 per cent. Estimating, then, 1 per cent. at least as the loss of heat from the top of the vessel—a loss of course, which does not enter into account in the main conductivity experiments—there would seem a possibility of an absolute error of 1·5 per cent. in the calorimetry of the method. But it must be realised that the electrical test, though interesting, and, perhaps, satisfactory, cannot be regarded as a rigorous electrical calibration of the flow calorimetry employed—for the small percentage of heat lost from the top is an uncertain quantity.

5. *Conclusion.*

The value of the thermal conductivity of mercury—viz., $0{\cdot}0196_0$ C.G.S. units over the range 35°C. to 45°C.—may be compared with the values obtained by other methods which are given in the table below :—

TABLE X.

Experimenter.	Method.	Temp.	Value of K.
H. F. Weber, 1880	Flat plate	17°C.	0·0162
Angström, 1864 ...	Periodic flow of heat	50°C.	0·0177
Berget, 1888.....	Ordinary guard-ring	0°C.-100°C	0·0202
R. Weber, 1903 ...	Electrical measurement of heat	0°C.-34°C.	0·0197
The Author, 1913	Impressed velocity	15·5°C.	0·0201

The present result is thus confirmatory of the high values obtained by "steady state" methods, as against the much lower, but, perhaps, more usually accepted values obtained by the "variable state" methods of Angström and H. F. Weber.

The author's belief is that the thermal conductivity of mercury is not far below the value 0·020, and, if anything, slightly increases with the temperature between, say, 30°C. and 50°C. In view of the importance attached by such authorities as Lord Kelvin and Callendar to periodic flow methods, and in view of the fact that Angström's experiments were performed as far back as 1864, prior, that is to say, to the development of electrical thermometry of precision, it might seem desirable to repeat some of the older work, introducing modern improvements.

Incidentally, the vacuum guard-ring has proved a success, and the accuracy with which temperature gradients can be measured within it would seem to warrant the development of a system of calorimetry better suited for measuring heat supplied at so small a rate. The other possibilities of such a guard ring have already been mentioned.

The author would record his appreciation of the interest taken in his work by the Head of the Physics Department, Dr. A. Griffiths, who has been specially considerate in his endeavours to obtain for the Author so much of the apparatus needed.

ABSTRACT.

A specially constructed vacuum vessel heated at the top by steam and cooled at the bottom by flowing water, is used to find the thermal conductivity of mercury. The vacuum acts as a guard ring, which is at the same time not open to the well-known objection of communicating to the calorimeter a quantity of heat difficult to estimate. So efficient is the vacuum that the temperature gradient as measured by a single thermo-junction carried by a cathetometer is probably not in error to the extent of 1 part in 500.

The calorimetry is effected by the continuous-flow method on the lines suggested by Searle in his well-known laboratory apparatus for finding the thermal conductivity of copper. The rate of supply of heat, however, is only about 1 calorie per second, and the space in which it is desirable to collect it is necessarily somewhat limited. It is thus only after considerable trouble that the conditions have been obtained which yield a consistency of about 1 per cent. between experiments performed on different days or on the same day under varying rates of flow. The reliance to be placed on the calorimetry is greatly confirmed by a series of tests made with a specially-designed electric heater. The mean value obtained for the thermal conductivity of mercury in a set of 24 experiments is $0{\cdot}0196_0$ C.G.S. units over the range 35°C. to 45°C. The remarkable linear nature of the temperature gradient obtained within the vessel, the cross-section of which was very uniform, over the larger range of temperature 35°C. to 65°C. would indicate at least that there is no diminution of thermal conductivity with rise of temperature.

All temperature measurements were made with iron constantan

thermo-couples, special care being taken to eliminate local thermo-electric effects. A simple arrangement is described for keeping a "cold junction" during the time of an experiment constant to 0·01°C. at temperatures above the room.

DISCUSSION.

The AUTHOR has communicated the following reply to points raised by various Fellows with whom he discussed the apparatus after the meeting: In general thermo-couples made of wires taken off the same reel do not agree closely in thermo-E.M.F. But junctions made of pure charcoal, iron wire and constantan taken off the same reels, the junctions being welded, and all other joints being twisted with pliers and immediately painted, agree with remarkable closeness, if the usual precautions be taken to quell local thermo-electric effects. I have tested at least six such couples, including the two pairs used in the research, which agreed at the worst to 1 part in 650. At Dr. Griffiths' suggestion I calibrated prior to the meeting two more such couples between fixed points, obtaining to the nearest microvolt 1,427 and 1,428 microvolts between 0°C. and 32·383°C., the transition point of sodium sulphate and 4,424 and 4,426 microvolts respectively between 0°C. and 99·64°C. Dr Griffiths and others witnessed the definiteness of the balances on the Cambridge thermo electric potentiometer. I have just discovered that Palmer ("Phys. Review," XXI., 1905) records similar agreement. Unfortunately, constantan is a very uncertain material, being known under various names, such as advance, climax, eureka, the small change in percentage composition causing large effects in the thermo-electric properties; a previous specimen I tested gave 50 microvolts per degree Centigrade. I regret I can form no estimate of the small quantity of heat lost per second from the sides of the vacuum vessel. While the experiments on water greatly confirm the efficacy of the guard-ring no concrete conclusion can be drawn from these, for it is impossible to say how much of the slight fall in temperature gradient is due to emissivity, and how much due to the rise of thermal conductivity of water with fall of temperature.

X. *Practical Harmonic Analysis.* *By* ALEXANDER RUSSELL, *M.A., D.Sc.*

RECEIVED NOVEMBER 30, 1914.

Introductory.

IF the value of a function, $f(x)$, recur over successive intervals, λ, of the variable x, so that the equation $f(x+n\lambda)=f(x)$ is true for all integral values of n, then Fourier showed that we may write

$$f(x)=a_0+a_1\cos(2\pi/\lambda)x+a_2\cos 2(2\pi/\lambda)x+\ \ldots$$
$$+b_1\sin(2\pi/\lambda)x+b_2\sin 2(2\pi/\lambda)x+\ \ldots \quad (1)$$

where a_0, a_1, b_1, . . . are constants which he expressed in the form of definite integrals. [*See* (2) given below.] He pointed out* that these coefficients were as real constants of a periodic curve as, for instance, its area or its centroid. Even when we neglect the higher harmonics in (1) it is advisable to use the Fourier values of the coefficients, as it is known† that, assuming the ordinary law of probability, these values make the sum of the mean squares of the errors a minimum.

In the " Philosophical Magazine " for August, 1874, p. 95, J. O'Kinealy showed how the theorem (1) quoted above follows at once from the ordinary symbolical methods used in solving differential equations.

Taylor's theorem for the expansion of a function of two variables may be written as follows :—

$$f(x+\lambda)=\varepsilon^{\lambda\frac{\partial}{\partial x}}f(x),$$

where ε is the base of Napierian logarithms. Hence, we may write the equation $f(x+\lambda)=f(x)$ in the form

$$(\varepsilon^{\lambda\frac{\partial}{\partial x}}-1)f(x)=0.$$

Regarding this as a differential equation, the roots of the auxiliary equation are given by

$$\varepsilon^{\lambda p}=1,$$

where p has been written for $\partial/\partial x$. Now, by De Moivre's theorem,

$$\varepsilon^{\lambda(2m\pi/\lambda)\iota}=\cos 2m\pi+\iota\sin 2m\pi=1,$$

where ι stands for $\sqrt{-1}$. Hence $p=m(2\pi/\lambda)\iota$, and m can have

* Joseph Fourier, " Theory of Heat," Freeman's translation, p. 198.
† C. H. Lees, " Proc." Phys. Soc., London, Vol. XXVI., p. 275, 1914

any positive or negative integral value. Therefore, by the rules given for solving differential equations, the general solution is

$$f(x)=a_0+a_1'\varepsilon^{(2\pi/\lambda)x\sqrt{-1}}+a_2'\varepsilon^{2(2\pi/\lambda)x\sqrt{-1}}+\ \ldots\ :$$
$$+b_1'\varepsilon^{-(2\pi/\lambda)x\sqrt{-1}}+b_2'\varepsilon^{-2(2\pi/\lambda)x\sqrt{-1}}+\ \ldots$$
$$=a_0+a_1\cos\,(2\pi/\lambda)x+a_2\cos\,2(2\pi/\lambda)x+\ \ldots$$
$$+b_1\sin\,(2\pi/\lambda)x+b_1\sin\,2(2\pi/\lambda)x+\ \ldots$$

And if $f(x)$ be real, the constants are all real. It will be seen that this is Fourier's theorem. Fourier also gave the following exact mathematical values for these constants :—

$$a_0=\frac{1}{\lambda}\int_0^{\lambda}f(x)\partial x\,;\ a_n=\frac{2}{\lambda}\int_0^{\lambda}f(x)\cos n\left(\frac{2\pi x}{\lambda}\right)\partial x\,;$$

and
$$b_n=\frac{2}{\lambda}\int_0^{\lambda}f(x)\sin n\left(\frac{2\pi x}{\lambda}\right)\partial x. \quad \ldots\ldots\ldots \quad (2)$$

The integral for b_n was previously given by Lagrange.* When we know the mathematical expression for $f(x)$, approximate numerical values of the constants can generally be determined without much difficulty.

In certain cases when the function is discontinuous difficulties arise in interpreting the meaning of the results obtained at the points where discontinuities occur, but this, although a point of great interest to the mathematician, does not affect the practical usefulness of the theorem and so we do not discuss it here.

In engineering practice we are, as a rule, given the graph of $f(x)$, and we have to determine the constants a_0, a_1, b_1, . . . The value of a_0 can be found at once by finding the area $\int_0^{\lambda}f(x)\partial x$ and dividing by λ. The area must be determined by some method of mechanical quadrature, several of which are in everyday use. It is to be noticed that this gives us an approximate value of the integral. In one method a certain number of ordinates are measured and the integral is expressed in terms of them. In exactly the same way we can find the values of a_n and b_n in terms of certain selected ordinates of the curves

$$y=f(x)\cos\,n(2\pi/\lambda)x \text{ and } y=f(x)\sin\,n(2\pi/\lambda)x$$

respectively, it being unnecessary to draw the graphs of these

* *See* Todhunter's "Integral Calculus," Chap. XIII.

curves, as the values of the ordinates can be found at once by multiplying $f(x)$ by the required cosine or sine coefficient. Before showing the best way of doing this, we shall give a brief *résumé* of the methods ordinarily employed.

Practical Methods.

There are three typical methods used in everyday work. In the first method all harmonics the order of which is higher than some value n are neglected. In this case the equation $y=f(x)$ contains $2n+1$ constants, and so we require to know the values of at least $2n+1$ ordinates in order to determine the constants. We get $2n+1$ equations of the form

$$y_1=a_0+a_1\cos(2\pi/\lambda)x_1+a_2\cos 2(2\pi/\lambda)x_1\ \ldots$$
$$+b_1\sin(2\pi/\lambda)x_1+b_2\sin 2(2\pi/\lambda)x_1\ \ldots$$

And many methods* have been given for lightening the labour involved in solving these equations for a_0, $a_1, b_1, \ldots$ and so finding the equation to a curve $y=f(x)$ having the same $2n+1$ ordinates as the given curve.

C. F. Gauss ("Werke," Vol. III., p. 281) showed how we could write down the required equation at once in the following form

$$y=y_1\frac{\sin\frac{1}{2}(x-x_2)\sin\frac{1}{2}(x-x_2)\ \ldots\ \sin\frac{1}{2}(x-x_{2n+1})}{\sin\frac{1}{2}(x_1-x_2)\sin\frac{1}{2}(x_1-x_3)\ \ldots\ \sin\frac{1}{2}(x_1-x_{2n+1})}$$
$$+y_2\frac{\sin\frac{1}{2}(x-x_1)\sin\frac{1}{2}(x-x_3)\ \ldots\ \sin\frac{1}{2}(x-x_{2n+1})}{\sin\frac{1}{2}(x_2-x_1)\sin\frac{1}{2}(x_2-x_3)\ \ldots\ \sin\frac{1}{2}(x_2-x_{2n+1})}$$
$$+\ \ldots\ \ldots\ \ldots

indication as to how far the calculated values of the harmonics differ from their true Fourier values. If n is large it is probable that the values of a_m and b_m found in this way are approximately correct when m is small; but if m be equal or nearly equal to n it is highly probable that the values are quite different from the Fourier values.

In the second method an attempt is made to determine the values of the definite integrals given in (2) by mechanical quadrature. For this purpose we have found that Weddle's rule, which we discuss below, is the most suitable. One of its advantages is that the value of the amplitude and phase of any given harmonic can be determined separately from the other harmonics to any required degree of accuracy. The lower the order of the harmonic the less is the arithmetical labour involved in finding its value.

In the third method certain infinite series are obtained connecting the values of the Fourier constants, the orders of which are odd multiples of n. When the harmonics diminish rapidly in value this method is a simple and easy one.*

We show below how to obtain many other series of a like nature, the use of which extends the range and increases the accuracy of the method.

The final conclusion we arrive at is that the method of evaluating the Fourier integrals by mechanical quadrature is by far the best. The series formulæ, however, are of value, especially as supplementary aids in checking the computed results.

Weddle's Rule.

The rule given by Thomas Weddle† for finding the area A included between a curve, two ordinates and the axis of x ($BPQN_6O$ in Fig. 1) is

$$A=\int_0^x y\delta x=\frac{x}{20}[y_0+y_2+y_4+y_6+5(y_1+y_5)+6y_3] \quad . \quad (4)$$

where $ON_1=N_1N_2=\ldots=N_5N_6=x/6$, and y_0, y_1, . . . are the values of the ordinates at the points O, N_1, . . .

* Fischer-Hinnen, "Elektrotechnik und Maschinenbau," Vol. XXVII., p. 335, 1909. S. P. Thompson, "Proc." Phys. Soc., Vol. XXIII., p. 334, 1911.

† "On a New and Simple Rule for Approximating to the Area of a Figure by means of seven Equidistant Ordinates," "The Cambridge and Dublin Mathematical Journal," p. 79, February, 1854.

The formula was obtained by the method of finite differences. The following proof, however, is simpler, more accurate and more instructive.

First of all the assumption is made that the equation to the curve can be put in the form

$$y=a_0+a_1x+a_2x^2+\ldots+a_nx^n \quad \ldots \quad (5)$$

This curve can be made to pass through $n+1$ points on the curve BPQ (Fig. 1), and it is the simplest curve that can be

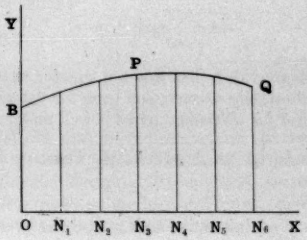


FIG. 1.—WEDDLE'S RULE.

drawn through these $n+1$ points. Its equation may be written in the form

$$y=y_1\frac{(x-x_2)(x-x_3)\ldots(x-x_{n+1})}{(x_1-x_2)(x_1-x_3)\ldots(x_1-x_{n+1})}$$
$$+y_2\frac{(x-x_1)(x-x_3)\ldots(x-x_{n+1})}{(x_2-x_1)(x_2-x_3)\ldots(x_2-x_{n+1})}\ldots \quad (6)$$
$$+\ldots$$

By expanding the coefficients of $y_1, y_2, \ldots$ in this equation and comparing with (5) the values of $a_0, a_1, a_2, \ldots$ are at once found.

By applying Weddle's rule to (5) we find that the term a_mx^m adds to the value found for the area the expression

$$\frac{a_mx^{m+1}}{20.6^m}[2^m+4^m+6^m+5(1+5^m)+6.3^m] \quad \ldots \quad (7)$$

and the term a_0 adds a_0x to the value of the area. Hence,

putting $m=1, 2, \ldots$ in (7) and simplifying we find that Weddle's rule gives the following value for the area :—

$$\begin{aligned} A = {} & a_0x + a_1(x^2/2) + a_2(x^3/3) + a_3(x^4/4) + a_4(x^5/5) \\ & + a_5(x^6/6) + a_6(x^7/7)(1{\cdot}00013) \\ & + a_7(x^8/8)(1{\cdot}00051) + a_8(x^9/9)(1{\cdot}00145) \\ & + a_9(x^{10}/10)(1{\cdot}00343) + a_{10}(x^{11}/11)(1{\cdot}00691) + \ldots \quad (8) \end{aligned}$$

By the integral calculus the true value of the area is

$$A = \sum_{p=0}^{p=n} a_p x^{p+1}/(p+1).$$

We see that, provided that n is not greater than 5, Weddle's formula is absolutely correct, and even for values of n as great as 10 the error *for individual terms* is well under 1 per cent.

For example, let us determine the value of $\int_0^{0{\cdot}6} \sinh x \partial x$ by (4). We get

$$\begin{aligned} \int_0^{0{\cdot}6} \sinh x \partial x = \frac{3}{100} & [\sinh(0{\cdot}2) + \sinh(0{\cdot}4) + \sinh(0{\cdot}6) \\ & + 5\{\sinh(0{\cdot}1) + \sinh(0{\cdot}5)\} + 6\sinh(0{\cdot}3)] \\ = {} & 0{\cdot}185\ 465\ 2\ \ldots \end{aligned}$$

which is correct to the last figure. On the other hand, we see by (8) that the error in the value of $\int_0^x x^{10} \partial x$ found by quadrature is 0·69 of 1 per cent. It is advisable, therefore, to have means of checking the computed values.

Series Formulæ Involving Areas.

It is easy to show by trigonometry that

$$\begin{aligned} \cos x - \tfrac{1}{3}\cos 3x + \tfrac{1}{5}\cos 5x - \ldots & \\ = \pi/4 \text{ from } x = (2n - 1/2)\pi \text{ to } (2n + 1/2)\pi & \\ = -\pi/4 \text{ from } x = (2n + 1/2)\pi \text{ to } (2n + 3/2)\pi, & \end{aligned}$$

and that

$$\begin{aligned} \sin x + \tfrac{1}{3}\sin 3x + \tfrac{1}{5}\sin 5x + \ldots = \pi/4, \text{from } x = 2n\pi \text{ to } (2n+1)\pi & \\ = -\pi/4, \text{from } x = (2n+1)\pi \text{ to } (2n+2)\pi. & \end{aligned}$$

Hence, by Fourier's formulæ (2), we find that

$$a_1-\frac{a_3}{3}+\frac{a_5}{5}-\ldots=\frac{2}{\lambda}\int_0^{\lambda}f(x)\left\{\cos\frac{2\pi x}{\lambda}-\tfrac{1}{3}\cos 3\frac{2\pi x}{\lambda}+\ldots\right\}\partial x$$

$$=\frac{\pi}{2\lambda}\left[\int_0^{\lambda/4}-\int_{\lambda/4}^{3\lambda/4}+\int_{3\lambda/4}^{\lambda}\right]y\partial x.$$

The series, therefore, on the left-hand side equals $(\pi/2\lambda)$ times the difference between the sum of the areas of the curve from 0 to $\lambda/4$ and from $3\lambda/4$ to λ, and the area of the curve from $\lambda/4$ to $3\lambda/4$.

Similarly, we see that

$$a_m-\frac{a_{3m}}{3}+\frac{a_{5m}}{5}-\ldots=\frac{\pi}{2\lambda}\left[\int_0^{\lambda/4m}-\int_{\lambda/4m}^{3\lambda/4m}+\int_{3\lambda/4m}^{5\lambda/4m}-\ldots\right]y\partial x \quad (9)$$

and $$b_m+\frac{b_{3m}}{3}+\frac{b_{5m}}{5}+\ldots=\frac{\pi}{2\lambda}\left[\int_0^{\lambda/2m}-\int_{\lambda/2m}^{3\lambda/2m}+\int_{2\lambda/2m}^{5\lambda/2m}-\ldots\right]y\partial x. \quad (10)$$

It will be noticed that the expressions on the right-hand side of (9) and (10) represent areas which can be found by any of the ordinary methods. Formulæ (9) and (10), therefore, enable us to find approximate values for the Fourier coefficients when the amplitudes of the higher harmonics are small.

Let us first consider the case of the rectangular wave shown

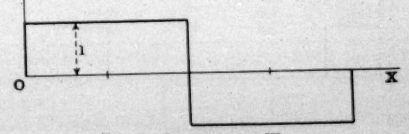


FIG. 2.—RECTANGULAR WAVE.

in Fig. 2. In this case y is $+1$, when x lies between 0 and $\lambda/2$, and -1 when x lies between $\lambda/2$ and λ.

By (9) we find that

$$a_m-a_{3m}/3+a_{5m}/5-\ldots=0,$$

for all values of m, and thus the coefficients of all the cosine terms are zero. By (10) we find that

$$b_m+b_{3m}/3+b_{5m}/5+\ldots=\pi/(2m), \text{ or } 0,$$

according as m is odd or even.

If we neglect all harmonics whose orders are higher than 16, we have

$$b_1+b_3/3+b_5/5+\ldots=\pi/2$$
$$b_3+b_9/3+b_{15}/5+\ldots=\pi/6$$
$$b_5+b_{15}/3+\ldots=\pi/10,$$
$$b_7+\ldots=\pi/14, \text{ \&c.}$$

And hence we easily compute the values of $b_1, b_3, \ldots$

Harmonics.	Computed values.	True values.	Harmonics.	Computed values.	True values.
b_1	1·286	1·273	b_9	0·174	0·141
b_3	0·445	0·424	b_{11}	0·143	0·116
b_5	0·279	0·255	b_{13}	0·121	0·098
b_7	0·224	0·182	b_{15}	0·105	0·085

The true values of $b_1, b_3, \ldots$ have been computed from the coefficients of the sine terms in the following well-known equation to the wave shown in Fig. 2—

$$y=\frac{4}{\pi}\left[\sin x+\frac{1}{3}\sin 3x+\frac{1}{5}\sin 5x+\ldots\right],$$

the wave-length being 2π.

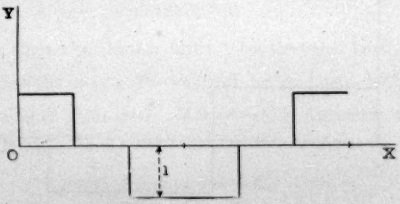


Fig. 3.—Discontinuous Rectangular Wave.

As the true values of the areas have been substituted in the formulæ used, the errors are due to the neglect of the seventeenth and higher harmonics. It will be seen that the error in the value of b_1 is about 1 per cent., and the errors in the computed values of b_7 and higher harmonics are greater than 23 per cent.

As a further example, let us consider the hightly irregular wave shown in Fig. 3.

The equation to this wave is

$$y=\frac{2\sqrt{3}}{\pi}\left[\cos x-\frac{1}{5}\cos 5x+\frac{1}{7}\cos 7x-\frac{1}{11}\cos 11x+\ldots\right].$$

In this case

$$a_1-\frac{a_3}{3}+\frac{a_5}{5}-\ldots=\tfrac{1}{4}\left[\int_0^{\pi/2}-\int_{\pi/2}^{3\pi/2}+\int_{3\pi/2}^{2\pi}\right]y\partial x,$$

$$=\int_0^{\pi/2}y\partial x=\frac{\pi}{3},$$

$$a_3-\frac{a_9}{3}+\frac{a_{15}}{5}-\ldots=0$$

$$a_5-\frac{a_{15}}{3}+\frac{a_{25}}{5}-\ldots=-\frac{\pi}{15}$$

$$a_7-\frac{a_{21}}{3}+\frac{a_{35}}{5}-\ldots=\frac{\pi}{21},\ \&c.$$

We also have $b_m+b_{3m}/3+b_{5m}/5+\ldots=0$ for all values of m. From the equations given above we see that $b_m=0$. Neglecting a_m when m is greater than 24, we easily compute the numbers given in the following table :—

Harmonics.	Computed values.	True value.	Harmonics.	Computed values.	True values.
a_1	1·100	1·103	a_{13}	0·081	0·085
$-a_5$	0·209	0·221	$-a_{17}$	0·062	0·065
a_7	0·150	0·158	a_{19}	0·055	0·058
$-a_{11}$	0·095	0·100	$-a_{23}$	0·046	0·048

The error in the value of a_1 is about 0·3 of 1 per cent., and in the value of the higher harmonics, which do not vanish, about 5 per cent.

Series Formulæ involving Ordinates (Thompson's Method.)

We have from (2)—

$$a_1+a_3+a_5+\ldots=\frac{2}{\lambda}\int_0^{\lambda}f(x)\left\{\cos\frac{2\pi x}{\lambda}+\cos 3\frac{2\pi x}{\lambda}+\ldots\right\}\partial x,$$

and hence, integrating by parts, we get

$$a_1+a_3+a_5+\ldots=\frac{1}{\pi}\left[f(x)\left\{\sin\frac{2\pi x}{\lambda}+\tfrac{1}{3}\sin 3\frac{2\pi x}{\lambda}+\ldots\right\}\right]_0^{\lambda}$$

$$-\frac{1}{\pi}\int_0^{\lambda}f'(x)\left[\sin\frac{2\pi x}{\lambda}+\tfrac{1}{3}\sin 3\frac{2\pi x}{\lambda}+\ldots\right]\partial x.$$

Noticing that the first expression on the right hand side vanishes at both limits, and that the series inside the bracket under the integral sign equals $\pi/4$ from 0 to $\lambda/2$ and $-\pi/4$ from $\lambda/2$ to λ, we get

$$\begin{aligned} a_1+a_3+a_5+\ldots &= -(1/4)[f(\lambda/2)-f(0)-f(\lambda)+f(\lambda/2)], \\ &= (1/2)[f(0)-f(\lambda/2)], \\ &= (1/2)(y_0-y_{\lambda/2}). \end{aligned}$$

In general, we have

$$a_m+a_{3m}+a_{5m}+\ldots = (1/2m)[y_0-y_{\lambda/2m}+y_{2\lambda/2m}-\ldots-y_{(2m-1)\lambda/2m}] \quad (11)$$

and

$$b_m-b_{3m}+b_{5m}-\ldots = (1/2m)[y_{\lambda/4m}-y_{3\lambda/4m}+y_{5\lambda/4m}-\ldots-y_{(4m-1)\lambda/4m}] \quad (12)$$

To illustrate the use of formulæ (11) and (12) let us consider the rectangular wave shown in Fig. 2. By (11) we see that all the coefficients of the cosine terms are zero, and by (12) if we neglect, as formerly, the 17th and higher harmonics we get

$b_{15}=1/15$, $b_{13}=1/13$, $b_{11}=1/11$, $b_9=1/9$, $b_7=1/7$, $b_5-b_{15}=1/5$, $b_3-b_9+b_{15}=1/3$, and $b_1-b_3+b_5-\ldots-b_{15}=1$. Hence, we find the computed values in the following table :—

Harmonics.	Computed values.	True values.	Harmonics.	Computed values.	True values.
b_1	1·224	1·273	b_9	0·111	0·141
b_3	0·378	0·424	b_{11}	0·091	0·116
b_5	0·267	0·255	b_{13}	0·077	0·098
b_7	0·143	0·182	b_{15}	0·067	0·085

We see that the error in the computed value of the first harmonic due to the method is greater than 4 per cent., and in b_7 and higher harmonics it is greater than 21 per cent. The accuracy obtained, therefore, in computing the values of b_7, b_9.. by this method is in this case slightly greater than that obtained by the method of areas. In the case of b_7, for instance, this proves that the neglect of $-b_{21}+b_{35}-\ldots$ produces a smaller error than the neglect of $b_{21}/3+b_{35}/5+\ldots$, when using the method of areas. If consecutive terms are of opposite signs it will be seen that the method of areas would determine the amplitudes of all the harmonics much more accurately.

Let us now consider the wave shown in Fig. 3. In this case

the difficulty arises as to the value we are to assign to the ordinate when the abscissa has the values $\pi/3$, $2\pi/3$, $4\pi/3$. ... We see that if the value of x is a little less than $\pi/3$, y is 1, but if it is greater than $\pi/3$, y is 0. It would seem reasonable, therefore, to take $(0+1)/2$ as the value of y and a rigorous mathematical proof of this can be given.

From (12) we can show that $b_m=0$, and from (11) we get that $a_{3m}=0$, and

$$a_1+a_5+a_7+a_{11}+\ldots=1.$$

$a_5+a_{25}+\ldots=-1/5$, $a_7=1/7$, $a_{11}=-1/11$, $a_{13}=1/13$,...

Hence, neglecting the 25th and higher harmonics, we get the computed values in the following table :—

Harmonics.	Computed values.	True values.	Harmonics.	Computed values.	True values.
a_1	1·121	1·103	a_{13}	0·077	0·085
$-a_5$	0·200	0·221	$-a_{17}$	0·059	0·065
a_7	0·143	0·158	a_{19}	0·053	0·058
$-a_{11}$	0·091	0·100	$-a_{23}$	0·044	0·048

The error in the value of a_1 is about 2 per cent., and in the value of a_5 and the higher harmonics it is greater than 10 per cent.

Comparison of the Series Formulæ Methods.

The waves analysed above are discontinuous, and the analysis shows that there is an infinite series of harmonics, the amplitudes of which are smaller the higher their order. The amplitudes of the harmonics, however, diminish slowly as their order increases. In both the examples considered above the amplitude of the 101th harmonic, for instance, is about 1 per cent. of the amplitudes of the fundamental. They, therefore, put the series methods to a severe test. We shall now consider waves which, although they are discontinuous and have an infinite number of harmonics, yet approach roughly in shape to a sine wave.

Let us consider, for instance, the wave which has the trapezoidal shape shown in Fig. 4. If the length of the straight line forming the top of the wave be $\lambda/6$, it is easy to show by Fourier's method that the equation to the curve shown in Fig. 4 is

$$y=\frac{6\sqrt{3}}{\pi^2}\left[\sin\frac{2\pi x}{\lambda}-\frac{1}{5^2}\sin 5\frac{2\pi x}{\lambda}+\frac{1}{7^2}\sin 7\frac{2\pi x}{\lambda}-\ldots\right].$$

As the wave approximates in shape much more closely to a sine wave than the cases previously considered the amplitudes of the harmonics are smaller, and we shall, therefore, consider the effect that neglecting the 7th and higher harmonics has on the accuracy of our results.

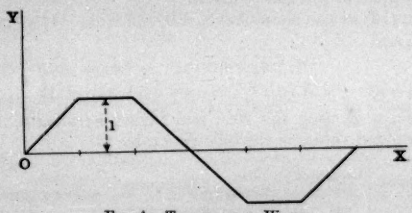


FIG. 4.—TRAPEZOIDAL WAVE.

Using the area method, we get by formulæ (9) and (10)

$$b_1+b_3/3+b_5/5=\pi/3\,;\quad b_3=0,\text{ and }b_5=-\pi/(3\cdot5^2).$$

Hence, $b_1=1\cdot056$ (1·053), $b_3=0$, $b_5=-0\cdot0419$ (−0·0421), the numbers in the brackets being the true values.

If we use the ordinates method, we get by (11) and (12)

$$b_1-b_3+b_5=1\,;\quad b_3=0\text{ and }b_5=-1/5^2.$$

Hence, $b_1=1\cdot040$ (1·053) and $b_5=-0\cdot040$ (−0·0421).

Both methods show that all coefficients the orders of which

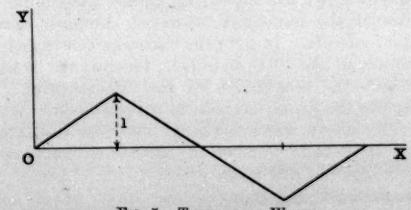


FIG. 5.—TRIANGULAR WAVE.

are multiples of 3 vanish. By taking higher harmonics into account the accuracy of the ordinates method could be made quite satisfactory in this case, but a very large number of ordinates would have to be measured with high accuracy.

As a further example, let us consider the triangular wave,

shown in Fig. 5. By Fourier's method we find that the equation to this wave is

$$y=\frac{8}{\pi^2}\left\{\sin\frac{2\pi x}{\lambda}-\frac{1}{3^2}\sin 3\frac{2\pi x}{\lambda}+\frac{1}{5^2}\sin 5\frac{2\pi x}{\lambda}-..\right\}.$$

The values of the ordinates and the areas required in formulæ (10) and (12) can easily be written down. The results of computing the harmonics by the two methods given above, when the 25th and higher harmonics are neglected are given in the following table :—

Harmonics.	True values.	Formula (10) (areas).	Formula (12) (ordinates).
b_1	0·8106	0·8106	0·8129
$-b_3$	0·0901	0·0901	0·0921
b_5	0·0324	0·0326	0·0356
$-b_7$	0·0165	0·0166	0·0181
b_9	0·0100	0·0097	0·0123
$-b_{11}$	0·0067	0·0065	0·0083
b_{13}	0·0048	0·0046	0·0059
$-b_{15}$	0·0036	0·0035	0·0044
b_{17}	0·0028	0·0027	0·0035
$-b_{19}$	0·0022	0·0022	0·0028
b_{21}	0·0018	0·0018	0·0023
$-b_{23}$	0·0015	0·0015	0·0019

In obtaining these results we assume that no less than 49 areas and 49 ordinates have been accurately measured. It will be seen that for all practical purposes the area method would be sufficiently accurate.

Accurate Formulæ.

As a general rule it is best to employ methods which aim directly at finding the values of the Fourier integrals. When the highest accuracy is desired it is essential to use these methods. One advantage they possess is that the amplitude and phase angle of each harmonic is determined independently of the others. In this case we can use the formulæ (9) to (12) to check the accuracy of our calculations.

To determine a_1 and b_1 accurately it is necessary to divide the wave-length λ into 24 equal parts, and measure the lengths of the ordinates $y_0, y_1, \ldots y_{23}$ at the ends and at the points of division, the ordinate y_{24} being equal to y_0, as the curve is periodic. By (2) we have

$$a_1=\frac{2}{\lambda}\int_0^{\lambda} y\cos\frac{2\pi x}{\lambda}\partial x,$$

$$=\frac{2}{\lambda}\left[\int_0^{\lambda/4}+\int_{\lambda/4}^{\lambda/2}+\int_{\lambda/2}^{3\lambda/4}+\int_{3\lambda/4}^{\lambda}\right]y\partial x.$$

Hence, applying Weddle's rule to each integral separately, we get

$$a_1=\frac{1}{40}\Big[y_0\cos 0+y_2\cos 2\frac{2\pi}{24}+y_4\cos 4\frac{2\pi}{24}+y_6\cos 6\frac{2\pi}{24}$$
$$+5\Big(y_1\cos\frac{2\pi}{24}+y_5\cos 5\frac{2\pi}{24}\Big)+6y_3\cos 3\frac{2\pi}{24}$$
$$+y_6\cos 6\frac{2\pi}{24}+y_8\cos 8\frac{2\pi}{24}+\ldots\Big]$$
$$=\frac{1}{80}[4y_0+y_4+y_{20}-(4y_{12}+y_8+y_{16})]$$
$$+\frac{\sqrt{3}}{80}[y_2+y_{22}-(y_{10}+y_{14})]$$
$$+\frac{1}{8}[\cos 15^0(y_1+y_{23}-y_{11}-y_{13})+\sin 15^0(y_5+y_{19}-y_7-y_{17})]$$
$$+\frac{3\sqrt{2}}{40}[y_3+y_{21}-y_9-y_{15}]. \quad \ldots\ldots\ldots (13)$$

Similarly, we find that

$$b_1=\frac{1}{80}[y_2+4y_6+y_{10}-(y_{14}+4y_{18}+y_{22})],$$
$$+\frac{\sqrt{3}}{80}[y_4+y_8-(y_{16}+y_{20})],$$
$$+\frac{1}{8}[\sin 15^0(y_1+y_{11}-y_{13}-y_{23})+\cos 15^0(y_5+y_7-y_{17}-y_{19})$$
$$+\frac{3\sqrt{2}}{40}[y_3+y_9-y_{15}-y_{21}]. \quad \ldots\ldots\ldots (14)$$

Applying these formulæ to the rectangular wave shown in Fig. 2, we get

$$a_1=0,$$
$$b_1=\frac{12}{80}+\frac{\sqrt{3}}{20}+\tfrac{1}{2}(\sin 15^0+\cos 15^0)+\frac{3\sqrt{2}}{10}$$
$$=1{\cdot}27324,$$

which is correct to the last figure.

The accuracy of the formula is thus of a high order. In everyday work we write $\sin 15^0=0{\cdot}259$ and $\cos 15^0=0{\cdot}966$ in (13) and (14).

Similarly, taking the same 24 ordinates, we get

$$a_2=\frac{1}{80}[4(y_0+y_{12}-y_6-y_{18})+y_2+y_{10}+y_{14}+y_{22}-(y_4+y_8+y_{16}+y_{20})],$$
$$+\frac{\sqrt{3}}{16}[y_1+y_{11}+y_{13}+y_{23}-(y_5+y_7+y_{17}+y_{19})] \quad . \quad (15)$$

$$b_2=\frac{\sqrt{3}}{80}[y_2+y_4+y_{14}+y_{16}-(y_8+y_{10}+y_{20}+y_{22})]$$
$$+\frac{1}{16}[y_1+y_5+y_{13}+y_{17}-(y_7+y_{11}+y_{19}+y_{23})]$$
$$+\frac{3}{20}[y_3+y_{15}-(y_9+y_{21})]. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

$$a_3=\frac{1}{40}[2y_0+y_8+y_{16}-(y_4+2y_{12}+y_{20})]$$
$$+\frac{\sqrt{2}}{16}[y_1+y_7+y_{17}+y_{23}-(y_5+y_{11}+y_{13}+y_{19})]$$
$$+\frac{3\sqrt{2}}{40}[y_3+y_{15}-(y_9+y_{21})]. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (17)$$

and

$$b_3=\frac{1}{40}[y_2+y_{10}+2y_{18}-(2y_6+y_{14}+y_{22})]$$
$$+\frac{\sqrt{2}}{16}[y_1+y_{11}+y_{17}+y_{19}-(y_5+y_7+y_{13}+y_{23})]$$
$$+\frac{3\sqrt{2}}{40}[y_3+y_9-(y_{15}+y_{21})]. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18)$$

To obtain the same accuracy for a_n as that given by formula (13) for a_1 we should have to measure 24 n ordinates. We should expect the accuracy of (17), for instance, not to be as high as that of (13), as we have taken the same number of ordinates in the two cases. Applying (17) and (18) to the rectangular wave (Fig. 2), we find that

$$a_3=0,$$
$$b_3=\frac{1}{40}(0)+\frac{\sqrt{2}}{16}(0)+\frac{3\sqrt{2}}{10}=0{\cdot}4243,$$

the true value being 0·4244.

The formulæ for the higher harmonics can be written down without difficulty, but naturally they are lengthy.

As a further example let us take the case of the circular wave, shown in Fig. 6. The wave-length is taken equal to 2.

If we only take 12 ordinates over the whole wave-length, then by Weddle's rule, owing to the symmetry of the wave,

$$10b_1=5y_{\lambda/12}+\sqrt{3}y_{2\lambda/12}+6y_{3\lambda/12},$$

and thus $b_1=0{\cdot}5680$. If we take 24 ordinates we get $b_1=0{\cdot}56703$, and, finally, if we take 36 ordinates we get $b_1=0{\cdot}56703$. This, therefore, is the true value of b_1. In this

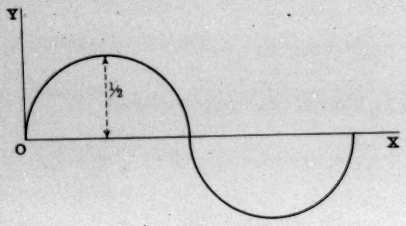


Fig 6.—Circular Wave.

case it is easier to get the true value of the Fourier integral by the approximate mechanical rule than by analysis.

Special Formulæ.

In certain cases series formulæ for determining the harmonics become very simple.

In alternating-current work, for instance, the positive half of the wave is generally exactly similar to the negative half, and so $a_0=0$, and all the even harmonics vanish. If, in addition, $f(x)=f(\lambda/2-x)$, formula (1) becomes

$$y=f(x)=b_1\sin(2\pi/\lambda)x+b_3\sin 3(2\pi/\lambda)x+\ldots$$

Hence we easily prove that

$$b_1-b_{11}+b_{13}-b_{23}+\ldots+b_{12n-11}-b_{12n-1}+\ldots$$
$$=(y_{\lambda/12}+y_{\lambda/4})/3+y_{\lambda/6}/\sqrt{3} \quad \ldots\ldots \quad (19)$$

and

$$b_5-b_7+\ldots+b_{12n-7}-b_{12n-5}+\ldots$$
$$=(y_{\lambda/12}+y_{\lambda/4})/3+y_{\lambda/6}/\sqrt{3} \quad \ldots\ldots \quad (20)$$

Similarly, we can show that

$$b_1+\frac{b_{11}}{11}+\frac{b_{13}}{13}+\ \ldots\ +\frac{b_{12n-11}}{12n-11}+\frac{b_{12n-1}}{12n-1}+\ \ldots$$
$$=\frac{2\pi}{3\lambda}\left[2\int_0^{\lambda/4}-\int_0^{\lambda/6}\right]y\partial x+\frac{2\pi}{\lambda\sqrt{3}}\left[\int_0^{\lambda/4}-\int_0^{\lambda/12}\right]y\partial x\ .\ .\quad(21)$$

and

$$\frac{b_5}{5}+\frac{b_7}{7}+\frac{b_{17}}{17}+\frac{b_{19}}{19}+\ \ldots$$
$$=\frac{2\pi}{3\lambda}\left[2\int_0^{\lambda/4}-\int_0^{\lambda/6}\right]y\partial x-\frac{2\pi}{\lambda\sqrt{3}}\left[\int_0^{\lambda/4}-\int_0^{\lambda/12}\right]y\partial x\ .\quad(22)$$

We also have by (10) and (12)

$$b_3-b_9+b_{15}-\ \ldots\ =\tfrac{1}{3}\left[2y_{\lambda/12}-y_{\lambda/4}\right]\ .\ .\ .\quad(23)$$

and

$$b_3+\frac{b_9}{3}+\frac{b_{15}}{5}+\ \ldots\ =\frac{2\pi}{\lambda}\left[2\int_0^{\lambda/6}-\int_0^{\lambda/4}\right]y\partial x\ .\ (24)$$

These equations sometimes enable us to find $b_1, b_3, \ldots$ with very little trouble. For example, in the case of the trapezoidal wave (Fig. 4), we see at once that $\lambda=2\pi$, $\int_0^{\lambda/4}y\partial x=\pi/3$, $\int_0^{\lambda/6}y\partial x=\pi/6$ and $\int_0^{\lambda/12}y\partial x=\pi/24$.

Hence, by (21),

$$b_1+b_{11}/11+\ \ldots\ =1{\cdot}053.$$

And thus neglecting small fractions of the amplitudes of higher harmonics of the orders $12n-1$ and $12n-11$ we get $b_1=1{\cdot}053$, which is correct to the last figure.

The equation to the circular wave (Fig. 6) is $y=\sqrt{x-x^2}$, and thus $y_{\lambda/12}=0{\cdot}3727$, $y_{\lambda/6}=0{\cdot}4714$ and $y_{\lambda/4}=0{\cdot}5000$. We also have

$$\int_0^{\lambda/2}y\partial x=0{\cdot}0430,\quad\int_0^{\lambda/6}y\partial x=0{\cdot}1146,\quad\text{and}\quad\int_0^{\lambda/4}y\partial x=0{\cdot}1964.$$

Hence, by measuring three ordinates and three areas only, we get by formulæ (19) to (24)

$b_1-b_{11}+b_{13}-\ \ldots$	$=0{\cdot}563$
$b_5-b_7+b_{17}-\ \ldots$	$=0{\cdot}019$
$b_1+b_{11}/11+b_{13}/13+\ \ldots$	$=0{\cdot}569$
$b_5/5+b_7/7+b_{17}/17+\ \ldots$	$=0{\cdot}013$
$b_1-b_3+b_5-\ \ldots$	$=0{\cdot}500$
$b_3-b_9+b_{15}-\ \ldots$	$=0{\cdot}088$

From these equations we might deduce as a first approximation that $b_1=0{\cdot}569$ (0·567), $b_3=0{\cdot}088$ (0·094), &c., where the true values are given in brackets. With the possible exception of b_1, we have no guarantee of the accuracy of our results, as the equations show that the higher harmonics are not negligible, and since for a circular wave $f'(x)=\infty$, when x is zero there must be an infinite number of terms in the Fourier series. The true values found by Weddle's rule are $b_1=0{\cdot}567$, $b_3=0{\cdot}0942$, $b_5=0{\cdot}0400$, $b_7=0{\cdot}0252$, . . . and as only ordinates need to be measured, the accuracy of the data is higher than when areas have to be measured.

As a final example, let us consider the curve $y=x-2x^3+x^5$ (Fig. 7), which is almost indistinguishable from the sine curve,

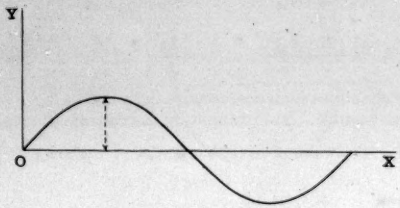


Fig. 7.—Biquadratic Wave.

whose equation is $y=(5/16)\sin\pi x$ for values of x between 0 and 1.

We have

$$y_{\lambda/4}=5/16,\ y_{\lambda/6}=22/81,\ \text{and}\ y_{\lambda/12}=\frac{205}{1296}.$$

Hence, if we neglect the 11th and higher harmonics, we get by (19)

$$b_1=(y_{\lambda/12}+y_{\lambda/4})/3-y_{\lambda/6}/\sqrt{3}=0{\cdot}313\ 704,$$

the true value being 0·313 705. We also have

$$\int_0^{\lambda/4} y\partial x=\pi/10,\ \int_0^{\lambda/6} y\partial x=61\pi/1215\ \text{and}\ \int_0^{\lambda/12} y\partial x=263\pi/19440$$

and thus, by (21),

$$b_1=0{\cdot}313\ 705.$$

By Fourier's method we can show that the equation to the curve is

$$y=\frac{96}{\pi^5}\left[\sin\frac{2\pi x}{\lambda}+\frac{1}{3^5}\sin 3\frac{2\pi x}{\lambda}+\ \ldots\right],$$

and so the values of the higher harmonics are extremely small. This explains the high accuracy attainable in this case by the series formulæ.

If the alternating-current wave be such that $f(x)=-f(\lambda/2-x)$, its equation must be of the form

$$y=f(x)=a_1\cos(2\pi/\lambda)x+a_3\cos 3(2\pi/\lambda)x+\ \ldots$$

In this case we have

$$a_1-\frac{a_{11}}{11}+\frac{a_{13}}{13}-\frac{a_{23}}{23}+\ \ldots=\frac{2\pi}{3\lambda}\left[\int_0^{\lambda/12}+\int_0^{\lambda/4}\right]y\partial x+\frac{2\pi}{\lambda\sqrt{3}}\left[\int_0^{\lambda/6}y\partial x\right] \quad . \quad . \quad (25)$$

and

$$\frac{a_5}{5}-\frac{a_7}{7}+\frac{a_{17}}{17}-\frac{a_{19}}{19}+\ \ldots=\frac{2\pi}{3\lambda}\left[\int_0^{\lambda/12}+\int_0^{\lambda/4}\right]y\partial x-\frac{2\pi}{\lambda\sqrt{3}}\left[\int_0^{\lambda/6}y\partial x\right]. \quad . \quad . \quad (26)$$

For instance, if we take the wave-form shown in Fig. 3 and neglect $a_{11}/11$, &c., we get

$$a_1=\frac{\pi}{6}+\frac{\pi}{9}\sqrt{3}=1{\cdot}13\ (1{\cdot}10),$$

and

$$\frac{a_5}{5}-\frac{a_7}{7}+\ \ldots=-0{\cdot}081\ (-0{\cdot}067),$$

where the true values are given in the brackets. The large errors in this case are due to the very distorted shape of the wave analysed. For waves approaching cosine shape the accuracy attainable would be far higher.

Conclusion.

The results given above prove that it is best to determine the values of the Fourier constants $a_0, a_1, a_2, \ldots b_1, b_2, \ldots$ of a periodic curve directly by mechanical quadrature. The

particular method of quadrature which the author has found best is that first suggested by Weddle. In a few cases, especially when the function has a point or points of discontinuity, the accuracy of the formulæ given is not of the highest order, although sufficient for practical work. In these cases, if higher accuracy is desired, it is necessary to subdivide the Fourier integrals, taking the limits at the points of discontinuity and finding the values of the component integrals separately by quadrature.

The various series formulæ given in the Paper sometimes enable us to find the values of the lower harmonics very readily. Their main use, however, lies in checking the values of the Fourier constants found by approximate formulæ.

ABSTRACT.

Making the assumption that the graph of a periodic function is given, the problem of the best way of determining the Fourier constants in the series equation which represents it is considered. The ordinary method of procedure is to neglect all the harmonics above a certain order and determine the coefficients of the harmonic terms by making the curve represented by this equation pass through a number of arbitrarily selected points on the given curve. This is the method used, for instance, by Runge and Grover. A serious defect in this solution is that the values found for the amplitudes of the harmonics, more especially for the higher harmonics, may be very different from their true Fourier values. The method gives no indication of the magnitude of these errors. Gauss pointed out many years ago that the solution of this limited problem could be written down at once mathematically, and that it was of importance in certain interpolation problems in astronomy. Another method has been suggested recently by Silvanus Thompson. He uses certain series formulæ for finding the Fourier constants. The author suggests other series formulæ of a similar kind. If the given curve be approximately sine shaped so that the amplitudes of the higher harmonics are small, this method is both simple and accurate. For distorted waves, however, the lack of accuracy is serious in practice. It has also the drawback that an error made in computing the value of one of the constants may introduce errors in the computed values of others. The author gives numerical examples to illustrate the accuracy attainable by the use of infinite series formulæ. He concludes by pointing out that in the great majority of cases much the best method of procedure when determining the constants is to evaluate Fourier's integrals by the methods of mechanical quadrature given in books on the calculus of finite differences. In particular he has found that Weddle's rule is admirably adapted for the practical computation of the Fourier integrals. By means of this rule, a new and simple proof of which is given, each constant is determined separately to a high order of accuracy. Numerical examples are given to illustrate this. The series formulæ used by Thompson can

be usefully employed either for verifying the values found by mechanical quadrature or for indicating when the higher harmonics cannot be neglected.

DISCUSSION.

The PRESIDENT thought the best way of impressing on students the importance of Fourier's theorem was to begin with the fundamental and add the successive harmonics one at a time, thus showing how the resultant curve approximates more and more nearly to the analysed original. In the problem treated by the Author the periods were selected arbitrarily. Very often it was necessary in practical problems to determine both the coefficients and the periods of harmonic constituents. The inverse process was performed in mechanical integrating machines, such as Kelvin's tide predicting machine, in which the curve resulting from the integration of a number of harmonic constituents of known amplitude and period was obtained.

Prof. S. P. THOMPSON thought any method which would give increased accuracy in the determination of the higher harmonics was welcome. He had suggested in a recent Paper that one should begin with the higher harmonics and work backwards towards the lower. This had the defect that any error in the initial determinations tended to accumulate. He wished to challenge the accuracy of Weddle's rule. He applied it to the case of an isoceles triangle, the length of which was three times the base. Dividing it into six parts and applying Weddle's rule, the area is given as 1·6 (the base being 1) instead of 1·5. He thought this rather discountenanced Weddle's rule, except for very smooth curves. It had long been desirable to have some harmonic analyser which would pick out a particular period without previous knowledge as the ear does in the case of sound. This was done to a certain extent in Schuster's periodogram.

Mr. F. J. W. WHIPPLE, criticising Prof. S. P. Thompson's example of the triangle, said that a triangle could not be regarded as the simplest curve through the six points chosen. If we have a smooth curve which is periodic, and apply Weddle's rule to the area included by a complete period divided by, say, six ordinates, we would get six separate values of the area if we start respectively at $y_0, y_1, y_2 \ldots$ Taking the mean of these we get the simple well-known expression

$$\tfrac{1}{2}(y_0+y_1+y_2+y_3+y_4+y).$$

He showed two slides showing a simple way by which an interpretation of a Fourier series, such as

$$\sin\theta+\sin 3\theta/3+\sin 5\theta/5 \ldots + \ \&c.$$

is easily obtained.

The AUTHOR, in reply, agreed with the President that one of the best ways of impressing the meaning of Fourier's theorem on the mind of the student was to construct graphically the curves formed by adding in succession the various harmonics to the fundamental, thus illustrating how the resulting curves approximated more and more to the shape of the original wave analysed. Michelson and Stratton have constructed a machine for drawing these curves automatically. It is described in the "Phil. Mag." for 1898, and many curves illustrating its use are shown. They show, for instance, the curves obtained by adding together three terms, five terms, seven terms, twenty-one terms and seventy-nine terms respectively of the Fourier series for a rectangular wave. The last figure they obtain is almost indistinguishable from a rectangle. He also referred to Arthur Wright's device for finding the harmonics electrically. Prof. Thompson points out that when Weddle's rule is applied to an isosceles triangle the error is nearly 7 per cent. The

reason of the large error is that the vertex of a triangular wave is a point of discontinuity. As mentioned in the Paper, therefore, Weddle's rule ought to be applied over each half of the base separately. When this is done the correct answer is obtained. Disappointment at the lack of accuracy sometimes obtained when this rule is applied indiscriminately is doubtless responsible for its neglect by many mathematicians, notwithstanding the high commendation passed on it by Prof. Boole nearly 60 years ago. It has a sounder theoretical basis than any similar rule, and in conjunction with the series formulæ given in the Paper it affords a method in many cases, possibly the only method, of computing the Fourier constants with high accuracy. The Author much appreciated Mr. Whipple's graphical method of showing how the sum of Fourier's sine series for $\pi/4$ gradually approximated to this value.

XI. *Measuring the Focal Length of a Photographic Lens.* By T. Smith, *B.A.*

Received November 26, 1914.

The principal focus of a lens of focal length f is at a distance fF/f' from that of the combination of focal length F formed by placing in front of the first lens another of focal length f'. This suggests a simple method of finding the focal length of a photographic lens, which can be divided into two parts, each capable of producing a real image of a distant object. Let f and f' be the focal lengths of the two components, and F that of the complete lens. Set up the whole lens in the camera, and focus a distant object sharply on the ground glass. Now unscrew the front component of the lens from its mount without disturbing the rest of the lens, and measure the distance d through which the ground glass has to be moved for the same object to be sharply focussed by the back component used alone. Then

$$d=\frac{f\mathrm{F}}{f'}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Next, take the whole lens out of the camera, and insert it the other way round, so that what is usually the back component is now in front. Focus as before with the complete lens for a distant object, and measure the displacement of the ground glass necessary to focus the same object when the component now in front is removed. Denote this distance by d'.

Then
$$d'=\frac{f'\mathrm{F}}{f}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

or, combining (1) and (2),

$$\mathrm{F}^2=d\,d'. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

This method avoids the difficulty of measuring exactly a transverse magnification, and also is not subject to errors arising from want of parallelism of object and image, from distortion and other oblique aberrations. When, as is often the case, the two components are equal, a single measurement suffices, the formula then reducing to

$$\mathrm{F}=d.$$

When the two components are unequal, the ratio of their focal lengths is determined by

$$f/f'=\sqrt{d/d'}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

The results (3) and (4) may be readily proved by noting that the positions in which the images are formed by the separate components are conjugate foci for the complete lens—viz., that pair of conjugate foci for which the beam of light between the two components is parallel.

When the separation of the two components of a lens combination can be increased by a known amount t, the focal length of either component can be found directly. Suppose that with this increased separation the focal length of the combination is F′, and the distance between its principal focus and that of the back combination alone is d''.

Then

$$\frac{1}{F}-\frac{1}{F'}=\frac{t}{ff'}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

and

$$d''=\frac{fF'}{f'}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

The elimination of F and F′ between (1), (5) and (6) gives

$$f^2\left(\frac{1}{d}-\frac{1}{d''}\right)=t \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

for finding the focal length of the back component. This result again is at once obvious from the ordinary expression for longitudinal magnification, since the principal foci of the front component and the focussing screen are in all cases in the positions of conjugate foci for the back component.

For the measurement of d'', when the lens mount has no means of adjustment, the front component can be conveniently fixed at one end of a short tube of metal or cardboard, the other end of which slips over or into the ordinary mount.

When f has been found, equation (4) gives f', the focal length of the front component.

A number of interesting variations suitable for exercising the ingenuity of a student will suggest themselves, *e.g.*, to find the focal lengths of a compound lens, such as the "Tessar," and of its components, one of which is diverging, when another converging lens of unknown focal length is provided as an auxiliary.

ABSTRACT.

The focal length of a compound lens is obtained solely by focussing on the camera screen the image of a distant object on the lens axis

by the complete lens and by each of its components separately. One additional focussing of the same object when the separation of the components is altered determines the focal lengths of each component. The method is both accurate and quick, and requires only a camera and the lens.

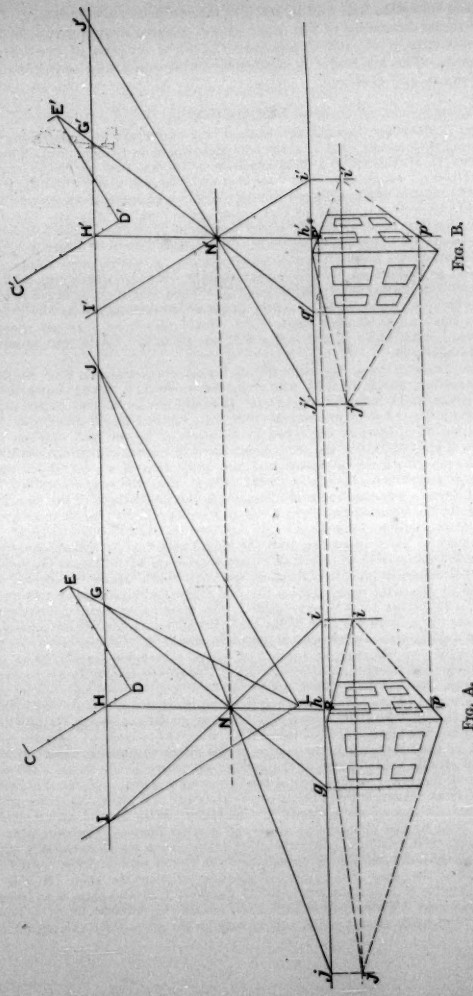
DISCUSSION.

Dr. A. Russell thought the method was extremely neat and likely to become very widely used by those who had such measurements to perform.

Prof. G. H. Bryan, in a communication which was read by the Secretary, described a "rough and ready" method which he had used to find the focal length of a lens when nothing was available but a photograph taken with the lens of some suitable object, such as a rectangular building (Fig. B). The two infinity points of perspective i' and j' are found by producing the horizontal lines of the building until they meet. The centre X of the picture is found, and through it a line is drawn perpendicular to $i'j'$, cutting it at a point, h'. On $i'j'$, as hypotenuse, a right-angled triangle, $i'N'j'$, is constructed, with its apex, N', on the perpendicular through X. Then, if X is on or near $i'j'$, $h'N'$ is the focal length of the lens. If the angle of the building is not 90°, but some other angle, θ, then angle $i'N'j'$ should also be θ. To get greater accuracy, with centre X and radius $h'N'$ cut $i'j'$ in Q. Qh' is then equal to the focal length.

The Author, in a partly communicated reply, commented on Prof. Bryan's construction, which, he said, was very neat so far as it went; but had the misfortune to be incorrect in general. It would not be difficult to find cases in which the result of measurements by this method would differ from the focal length of the lens employed by as much as 75 per cent. When the effect of the position of the stop, which limits the beam of light transmitted by the lens, upon the perspective of the projection of a solid object on a plane is adequately considered it will be seen that the length obtained by Prof. Bryan's construction is not necessarily the focal length of the lens, but is merely the distance from which the picture should be viewed to secure a correct impression of the view.

It may be noted in passing that the length obtained by this construction is liable to a number of errors which are frequently by no means negligible. Apart from those due to distortion, &c., the result would be affected by trimming the print unequally on the two sides, and considerable trimming is the rule rather than the exception. The more exact method mentioned appears to be incorrect even when Prof. Bryan's assumptions are adopted, for in taking such a photograph the plate would be vertical, the lens axis horizontal and the proper amount of subject included on the plate by employing the rising front of the camera. The perspective of the photograph of a solid object is determined, not by the focal length of the lens, but by the position of the stop which limits the light beams that are transmitted by the lens. In Fig. A let CDE represent in plan part of the exterior of the building of which a photograph is to be taken; let N be the position of the front nodal point of the lens, and L the centre of the effective limiting stop. If the actual stop is behind some of the component lenses L will be the (virtual) image of its centre in that part of the lens system which is situated in front of the stop. Assume that the lens gives an image of a plane object free from aberrations, and in particular from distortion, curvature and astigmatism. Let IHGJ be the plane focussed on and *ihgj* its image. The rays from a point, E, on the cottage, not in the plane focussed for, which eventually pass through the lens, fill a cone whose vertex is E and base a circle, centre L, parallel to the plane of the plate. This cone will meet the plane IHGJ in a circle, centre G, and what passes as the image of E on the plate is an image of this circle. If the stop is sufficiently small this circular image will be indistinguishable from a point, and we may for the purposes of this argument


J'
E'
G'
H'
D'
C'
I'
N'
i'
k'
p'
q'
J
E
G
H
D
C
I
N
i
k
p
q
j
Fig. B.
Fig. A.

consider only the principal ray of each beam of rays from any point of the object. All these principal rays will be directed towards L. So far as the lens is concerned and the image which it produces on the plate, each point E of the building may then be replaced by the point G, in which EL meets the plane focussed on. The photograph is a copy on a definite scale of the projection thus obtained on this plane, and may be determined by joining each point G to N and (taking in the usual way for graphical construction the two nodal points in coincidence) producing this straight line on to meet the image plane in g. The vanishing points I and J for horizontal lines in the directions CD and DE are obviously found by drawing LI and LJ parallel to these directions, meeting the focussed plane in I and J. The images i and j of these points determined by the usual construction are the vanishing points for the photograph.

In Fig. A the point L is shown much nearer to the plate than the nodal point N, and in Fig. B, where corresponding accented letters are used, the case is shown where L′ is in coincidence with the nodal point. In both diagrams the focal length of the lens is the same, and the positions of the principal planes are identical. The two pictures differ very much from one another in their perspective and in the values of the lengths obtained by the construction suggested. The only parts of the two pictures in which the dimensions are equal are those such as Pp, P′p′, in which the plane focussed for intersects the building. It is clear from these diagrams that the perspective, with a perfect lens such as is always assumed in considering elementary laws, depends on the position of the stop alone. If a lens of a different focal length were used to depict the same object, with its effective stop in the same position L, the perspective of the photograph would be exactly like that of Fig. A, though the two pictures would differ in size. Fig. B is the kind of case which Prof. Bryan has assumed to be general. This assumption is correct when the image is formed by a thin lens whose boundary is the limiting aperture, or when the lens is composed of two equal and similar components, with a stop placed symmetrically midway between them; but many lenses are by no means like this. As extreme forms we have telephoto combinations, which are simply systems in which the nodal point N is situated a considerable distance in front of L, and at the same time the lenses are kept of reasonable dimensions by constructing them of a converging system situated in the neighbourhood of L, followed by a diverging system between the converging system and the plate.

The foregoing considerations make it clear that Prof. Bryan's interesting construction cannot be relied on to give even a rough indication of the focal length of the lens by which a given photograph has been taken.

Prof. G. H. Bryan communicated the following note in reference to the points raised by the Author in his reply: Referring to Fig. A, let M be the point on HNh at which ij subtends a right angle in the case of a rectangular building (or in the more general case an angle equal to the angle between the faces of the building). Then Mh is the length F_1 which I take for the focal length of the lens, while, according to Mr. Smith's construction, the focal length F should be Nh. (Of course this is the focal length *as focussed on the object*, and differs from the true focal length by the distance the lens has been displaced from the infinity position.) Now, it will be seen from similar triangles that

$$F_1 : F = LH : NH.$$

If the camera is focussed on a building or other object (as implied in the words "suitable object" in my note) whose distance is large compared with the distance LN, F_1 will be very approximately equal to F. If the camera is focussed for parallel rays the construction will be exact, at any rate under the assumptions involved in Mr. Smith's arguments. When writing the note I never contemplated the possibility that the method would be applied in cases where the dimensions of the optical system were comparable with the

distance of the object to be photographed. Mr. Smith admits that the method would be correct if the shop were placed at the optical centre of the system. If this is not the case, distortion of the image *may* take place, and in such cases errors would undoubtedly occur. As regards the effect of unequal trimming, the error introduced will be small if the faces of the building make angles of about 45 deg. with the line of sight. But the object which I had in view in connection with this method was to ascertain how wide-angled a lens would be required to take in the whole of an architectural subject of which a photograph was available, and it is clear that, if the picture has been trimmed, the estimate will be more than sufficient. The accuracy of the method, of course, depends on the accuracy with which the infinity points can be constructed in the photograph, and consequently the method is limited to lenses which are not too narrow angled. I consider that the method is, therefore, correctly described in my note as a "rough-and-ready" method which is convenient for purposes such as those sketched out, for which Mr. Smith's methods would be unsuitable. A photographer using the method with camera in his own possession would naturally select a "suitable object," as suggested in my note, and would avoid using a rising front or trimming his print. In such cases it is impossible that the errors could approach anywhere near Mr. Smith's estimate, unless the lens system possessed excessive distortion. As regards the use of the method in connection with photographs taken under unknown conditions by other people, the method will certainly indicate conditions under which a similar photograph may be taken, and even in these cases it is difficult to see how the errors could possibly be so great as Mr. Smith estimates.

Mr. W. J. HALL (communicated) said that he had tried Prof. Bryan's method and found with a Cooke lens (nominal 5·5″ focal length (values 5·56″, 5·90″, 5·63″, 5·49″ and 5·50″ for the focal length, the distance between the lens and object being varied. With another lens of a different type, of which the focal length as determined by the very accurate method of Mr. Smith, was 14·60 cm., he obtained by Prof. Bryan's method 14·43 cm., 14·29 cm. and 14·50 cm. at different trials.

XII. *The Polyscope and its Projection.* *By* Prof. A. W. Bickerton, *A.R.S.M.*

Shown at the Meeting on January 22, 1915.

The instrument consists of three narrow strips of plate glass about a foot long, arranged as a truncated pyramid. The object end has an area of about a quarter of a square inch, while the eye end is made very small (about a twentieth square inch) in order to get the maximum number of reflections. To get geometrical accuracy in the inter-facial angles the edges of the strips overlap instead of being butted. Three types were shown. In No. 1 the cross-section is an equilateral triangle. This produces the simplest type of pattern. In No. 2 the section is a right-angled isosceles triangle. This produces patterns with two centres of symmetry of eight reflections each and one of four reflections. In the third type the angles are 30 deg., 60 deg. and 90 deg. This produces patterns in which one centre has 12 reflections, one six and one four. When a group of nondescript objects are viewed through the instrument, an exquisitely coloured symmetrical pattern may be seen the character of which changes continuously as the object is moved about.

Suitable slides can readily be constructed by mounting pieces of coloured gelatine, fragments of lace, bits of wire, &c., on glass.

The designs were shown projected on a screen, a narrow beam of light from an arc lamp being passed through the polyscope, which was moved about until the maximum illumination was obtained. The beam before reaching the object slide passed through a water tank and a ground glass plate to ensure uniform brightness.

The instrument forms an invaluable aid to designers of tiles, floorcloths, fabrics, &c., as an innumerable sequence of designs can be reviewed by the artist in a short time, those which give satisfaction being sketched or photographed. If the slide is suitably mounted and moved reasonably slowly it is quite easy to repeat any particular pattern.

XIII. *On the Criterion of Steel Suitable for Permanent Magnets.*

By SILVANUS P. THOMPSON, *D.Sc., F.R.S.*

RECEIVED JANUARY 19, 1915.

ERRATA

IN

VOL. XXVII. PART I.

On pag 94, line 35, for *iron* read *ion*.
„ „ 97 „ 24 „ *proportion* read *proportional*.
„ „ 100 „ 20 „ *induction* read *inductive*.
„ „ 101 „ 34 „ *unlike* read *unlikely*.
„ „ 111 „ 25, insert *pulse* after *Rontgen*.

The quantity $\sqrt{M\Omega}$ is in the denominator of all the expressions in which it occurs on page 95 *et seq*.

sufficiently great to enable a practical saturation to be attained during the magnetizing process—suffice to form a judgment as to its suitability for the purpose of making permanent magnets. A field of intensity $\mathfrak{H}=500$ is amply sufficient for any kind of steel; and for any but the very hardest a field of $\mathfrak{H}=250$ is adequate. Fig. 1 will make the matters at issue plain. It represents, for a fine quality of magnet steel, the descending branch of a hysteresis loop, the ordinates being values of $\mathfrak{J}$, the "magnetization" (magnetic moment per cubic centimetre) and the abscissæ the values of $\mathfrak{H}$. The value of $\mathfrak{J}_{rem.}$ the remanent magnetism (for a ring or a very long bar), is the ordinate OR, and the coercive force $\mathfrak{H}_c$ is the abscissa OC. In this steel $\mathfrak{J}_{rem.}=808$, and $\mathfrak{H}_c$ is 63. The lines which slope upward to the left from the origin represent graphically, by the respective tangents of the angles of their slope, the values (to a scale of $\frac{1}{16}$) of the coefficients of self-demagnetization for bars

XIII. *On the Criterion of Steel Suitable for Permanent Magnets.*

By SILVANUS P. THOMPSON, *D.Sc., F.R.S.*

RECEIVED JANUARY 19, 1915.

IN order that a steel shall be suitable for the making of permanent magnets, the prime requisites are that it shall have a large remanent magnetization and a high coercive force. These two qualities are independent of any question as to form. But it is well known that bars which are short have but a small remanent magnetization owing to their self-demagnetizing coefficient being great. In comparing together steels of differing composition, or steels of identical composition which have been subjected to different heat-treatments, it is therefore necessary to assume that the specimens to be compared are of similar form. According to all evidence to the present time, the coercive force of any given specimen of steel is independent of the form, provided that it has been subjected to a sufficiently great magnetizing force, the self-demagnetizing coefficient affecting the degree of remanent magnetization, but not affecting the degree of coercion with which that residuum is held by the steel.

When any specimen of steel is subjected to a cycle of magnetization so as to obtain its hysteresis loop, the dimensions of the loop—provided the maximum magnetizing force was sufficiently great to enable a practical saturation to be attained during the magnetizing process—suffice to form a judgment as to its suitability for the purpose of making permanent magnets. A field of intensity $\mathfrak{H}=500$ is amply sufficient for any kind of steel; and for any but the very hardest a field of $\mathfrak{H}=250$ is adequate. Fig. 1 will make the matters at issue plain. It represents, for a fine quality of magnet steel, the descending branch of a hysteresis loop, the ordinates being values of $\mathfrak{J}$, the "magnetization" (magnetic moment per cubic centimetre) and the abscissæ the values of $\mathfrak{H}$. The value of $\mathfrak{J}_{rem.}$ the remanent magnetism (for a ring or a very long bar), is the ordinate OR, and the coercive force $\mathfrak{H}_c$ is the abscissa OC. In this steel $\mathfrak{J}_{rem.}=808$, and $\mathfrak{H}_c$ is 63. The lines which slope upward to the left from the origin represent graphically, by the respective tangents of the angles of their slope, the values (to a scale of $\frac{1}{10}$) of the coefficients of self-demagnetization for bars

of about 10, 12, 15, 25 and 35 diameters' length respectively. The abscissa at the point P, where the sloping line crosses the curve, shows the residual magnetization for a bar that is 25 diameters long which has a coefficient of self-demagnetization of about 0·05. At this point the residual magnetization is reduced to about 640, and the demagnetizing force will be about $640 \times 0{\cdot}05 = 32$.

For all specimens of steel the hysteresis curves will be of the same generic form; but the two chief parameters OR and OC, that is $\mathfrak{J}_{rem.}$ and $\mathfrak{H}_c$, will differ. In the accompanying table* are given the values of $\mathfrak{J}_{rem.}$ and $\mathfrak{H}_c$, which have been found by various observers—Ewing, Du Bois and Taylor Jones, Mme. Curie and others—for a number of different brands of iron and

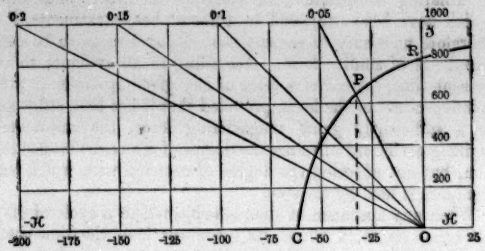


Fig. 1.

steel. The highest value found for $\mathfrak{J}_{rem.}$ is 900, in a specimen of unquenched Allevard steel (carbon content 0·59 per cent., tungsten content 5·5 per cent.); but it had only 26 for $\mathfrak{H}_c$. The highest value of $\mathfrak{H}_c$ is 85, for a molybdenum steel (carbon 1·25, molybdenum 3·36 per cent.), but its $\mathfrak{J}_{rem.}$ was only 370. No steel has yet been found which comes up to the quite possible ideal of having $\mathfrak{J}_{rem.} = 800$, with a coercive force of $\mathfrak{H}_c = 80$. The best magnet steel will have the values of both parameters high. For short magnets where the self-demagnetizing influences are great, it is more important that $\mathfrak{H}_c$ should be high, than that $\mathfrak{J}_{rem.}$ should be great.

* Originally given in the Author's Paper on the "Magnetism of Permanent Magnets," Journal of Institution of Electrical Engineers, Vol. L., p. 80, 1913.

Material.	H_c.	I_r.	$H_c \div I_r$.	$H_c \times I_r$.
Swedish wrought iron	0·8	550	0·00145	440
Softest selected iron	0·44	804	0·00547	322
Piano steel wire	22·0	824	0·02670	1,813
Piano steel wire	40·0	760	0·05260	30,400
Low-carbon steel	3·4	625	0·00544	1,900
High-carbon steel	58·0	645	0·09000	37,400
Haarlem magnet steel	56·0	800	0·07000	44,800
Allevard steel	26·0	900	0·02890	23,410
Allevard steel	73·0	800	0·09120	58,400
Böhler's Styrian steel	34·0	790	0·04300	26,860
Böhler's Styrian steel	75·0	600	0·12500	45,000
Remy's tungsten steel	63·0	808	0·07790	51,000
Remy's tungsten steel	77·0	800	0·09620	61,600
Medium tungsten steel	58·8	572	0·10300	33,600
Whitworth tungsten steel	37·0	610	0·06070	22,550
Molybdenum tungsten steel	85·0	370	0·23000	31,400
Chilled cast iron	52·5	218	0·24300	11,530
Lodestone	50·0	350	0·14300	17,500
High carbon steel, 1·2 per cent. C., quenched at 905°C.	48·0	264	0·18100	12,650
Cast iron (Hopkinson)	3·8	312	0·01230	1,190
Manganese steel annealed (Hopk.)	24·5	43	0·57200	1,054
Grey cast iron (Hopkinson)	13·67	250	0·05460	3,410
Chrome steel (Mathews, 7th)	52·5	690	0·07620	36,250
Tungsten steel (Mathews, 1st)	71·5	806	0·08870	57,700
Chrome steel tempered (Mathews, V., 3rd)	22·0	1,030	0·02135	22,660
Chrome steel (Miss Moir)	56·0	286	0·1954	16,016

Recently the question of the suitability of steels for permanent magnets has been discussed by Mr. J. A. Mathews, in the Proceedings of the American Society for Testing Materials, Vol. XIV., p. 51, 1914, who suggests that a new criterion of magnetic hardness should be framed upon the basis of the ratio between $\mathfrak{B}_{rem.}$ and $\mathfrak{H}_c$, or on the inverse ratio of $\mathfrak{H}_c \div \mathfrak{B}_{rem.}$, preferably the latter, as the magnetically harder steels then have the higher number. He proposes to call this ratio the "permanence."

Mr. J. R. Ashworth has made precisely the same suggestion in "Nature," January 7, 1915, p. 506, with the difference that he states the "permanence" in terms of the ratio $\mathfrak{H}_c \div \mathfrak{J}_{rem.}$. But, as $\mathfrak{B}_{rem.} = 4\pi\mathfrak{J}_{rem.}$, the distinction is one of scale only. Mr. Ashworth explicitly suggests this ratio as the criterion for "determining the most suitable material from which to construct a permanent magnet."

Now, ever since Hopkinson's determinations 30 years ago, it has been supposed that for purpose of making permanent magnets the best material would be that for which both $\mathfrak{J}_{rem.}$

and $\mathcal{H}_c$ should be high; and for which the product $\mathcal{H}_c \times \mathcal{J}_{rem.}$ was, therefore, a maximum.

To decide between this view and the new suggestion to take the ratio as criterion, instead of the product, the values of $\mathcal{H}_c \div \mathcal{J}_{rem.}$ and of $\mathcal{H}_c \times \mathcal{J}_{rem.}$ have been added to the table. The results are illuminating. Two of the materials—viz., molybdenum steel and chilled cast iron, are seen to have the ratio $\mathcal{H}_c \div \mathcal{J}_{rem.}$ nearly equal—viz., 0·23 and 0·243 respectively. But their products $\mathcal{H}_c \times \mathcal{J}_{rem.}$ are 31,400 and 11,530 respectively! And as molybdenum steel is 1·7 times higher than chilled cast-iron in the quantity of magnetism it retains, it follows that to produce two magnets of equal pull of these materials the cast-iron one must have $1{\cdot}7 \times 1{\cdot}7 = 2{\cdot}87$ times the cross-section of the molybdenum steel; and, even then, would have only 0·6 as great a coercive force! Or, again, according to the proposed criterion of Mr. Mathews and Mr. Ashworth, annealed manganese steel, which has a ratio of 0·572, should be six times as good as Remy tungsten steel, which has a ratio of 0·096. But the manganese steel has a remanence of only 43, compared with 800 for the tungsten steel—that is, 18½ times as little—and to make the two magnets of equal power the magnet of manganese steel would have to be constructed with a sectional area 314 times as great as that of the tungsten steel! The respective values of the product $\mathcal{H}_c \times \mathcal{J}_{rem.}$ for these two steels are 1,054 and 61,600 respectively; or, as about 1 to 60. If we have to choose a criterion between manganese steel, which is almost non-magnetic, and the Remy steel, which is one of the finest magnet steels known, we should choose the one which makes out that the Remy steel is 60 times as good as manganese steel, rather than that which pronounces manganese steel to be six times as good as Remy steel.

Very little is known about the magnetism of such substances as basalt and terra-cotta; but it is known that their remanent magnetization is very slight and their coercive force relatively enormous. But, because for them the ratio $\mathcal{H}_c \div \mathcal{J}_{rem.}$ probably exceeds that of any known brand of steel, no one would dream of recommending these as the most suitable materials for making permanent magnets.

It was suggested above that a possible ideal steel for permanent magnets is one which will have $\mathcal{J}_{rem.} = 800$ with $\mathcal{H}_c = 80$; here the ratio is 0·1 and the product is 64,000. I believe the highest $\mathcal{J}_{rem.}$ yet recorded is Mr. Mathews's value of $\mathcal{J}_{rem.} = 1{,}030$.

This was for a chrome steel that had $\mathfrak{H}_c$ only 22. The highest value of $\mathfrak{H}_c$ yet observed is Mme. Curie's figure of 85 for a molybdenum steel; but its $\mathfrak{J}_{rem.}$ was only 370. If it were possible for metallurgists to produce steel with $\mathfrak{J}_{rem.}$ 1,000 and $\mathfrak{H}_c$=100, the value of the ratio would still be only 0·1; but the product would then be 100,000, and for equal power the sections and weights might be reduced to about 64 per cent. of those of our best permanent magnets of to-day. Or, for given section and weight, magnets made of this ideal steel would have about 24 per cent. greater pull, and 20 per cent. greater coercive force.

ABSTRACT.

Whatever the form to be given to a permanent magnet, the prime requisites as to the quality of the steel are (1) large remanent magnetism ($\mathfrak{J}_{rem.}$) and a high coercive (force $\mathfrak{H}_c$). Since Hopkinson's determinations of 1885 it has been supposed that for the purpose of making permanent magnets the best material would be that for which both of these quantities and, consequently, their numerical product, should be as high as possible. Recently Mr. J. A. Mathews and, independently, Mr. J. R. Ashworth, have proposed to take the ratio $\mathfrak{H}_c \div \mathfrak{J}_{rem.}$ or $\mathfrak{H}_c \div \mathfrak{B}_{rem.}$ which only differs in scale, as the criterion.

To decide as to the suggestion to take the ratio instead of the product a table giving the values of $\mathfrak{H}_c$, $\mathfrak{J}_{rem.}$, $\mathfrak{H}_c \div \mathfrak{J}_{rem.}$ and $\mathfrak{H}_c \times \mathfrak{J}_{rem.}$ for a number of steels is given, and it is clearly shown that the use of the ratio as a criterion of magnetic usefulness leads to most absurd results. As an example, annealed manganese steel (almost non-magnetic) should, judging from the ratio, be six times as good as Remy steel, whereas in reality, for equality of pull, a magnet of manganese steel would require to be 312 times the weight of that made of Remy steel. The qualities requisite in an ideal steel for permanent magnets are indicated.

DISCUSSION.

Dr. S. W. J. Smith said that in cases (by far the most numerous) in which it was as essential that the moment of the magnet should be high as that it should be permanent, the criterion $H_c \div I_r$ a maximum obviously had very little to be said for it in comparison with the criterion $H_c \times I_r$ a maximum as Prof. Thompson had so clearly shown. It was possible, however, to think of cases where a "permanent magnet" might be required in which the constancy of the moment might be much more important than its magnitude. In these cases more might be said in favour of the first criterion. He pointed out that the choice between two different materials for a particular purpose might depend as much upon the shape of the IH curve as upon the values of I_r and H_c.

Mr. A. Campbell said that for practical purposes it was important that I_r should be large or else a large coercive force was of no use. He agreed therefore that the product of H_c and I_r was the most useful criterion, and could not think of a case where the quotient would give a useful indication of the quality of the steel.

Dr. C. Chree said that peculiar cases were often encountered in which it was difficult to say exactly what interpretation should be put on the terms "permanent magnetism," "coercive force," &c. Some years ago he had had occasion to demagnetise a bar of cobalt. He had used the ordinary method of reversals, accelerating the process by hammering. By this process he actually reversed the magnetisation, a phenomenon which he attributed to the original magnetism being in the opposite direction in the outer layers to what it was in the inner core, the magnetism of the outer layers being destroyed by the hammering. He had also found with a certain dip needle that it could be magnetised more strongly in one direction than in the other. It was afterwards found that the steel from which the dip needle was made had been magnetised very strongly in that direction by the maker. There was another factor of importance besides the permanence of the moment of a magnet, and that was permanence of the distribution of the magnetisation. He cited the case of two Kew magnetometer magnets which had inadvertently been allowed to touch. Their moments were unaltered, but their mutual influence was considerably affected owing to a change in the distribution of the magnetisation.

Mr. E. H. Rayner communicated the following: From a practical point of view a high value for the product of remanence × coercivity is to be desired with the proviso that a low value in one of them can not compensate for a corresponding unusually high value in the other. A "figure of merit" might be made out by penalising the material in both of these properties, counting only after a certain value has been attained. This figure might be calculated from the formula $(I_r - 400) \times (H_c - 20)$. This would give a positive figure of merit to 15 materials in the list given in the Paper, implying that from a practical point of view the rest were comparatively worthless. The order of those with a positive figure of merit is much the same as the order of the magnitudes given in the last column of the Paper, the first four being in the same order and the fifth and sixth inverted. The most notable exception is the molybdenum tungsten steel which comes tenth in order of merit in the last column of the Paper, but which by reason of having a value for I_r of less than 400—namely, 370—has a negative figure of merit on the basis suggested. It would be interesting if Prof. Thompson could give some idea of what is the minimum value for H_c which is desirable in a good permanent magnet for commercial purposes such as magnetos for explosion engines.

Mr. J. R. Ashworth communicated the following: Prof. Thompson has discussed two criteria of steel suitable for permanent magnets, but has only applied one test in deciding upon their merits—namely, the pull which the magnet can exert. On the other hand, when I suggested the ratio of H_c to I_r as the definition of "permanence," I had in view, as I distinctly stated, "the preservation of the magnetism in a magnet," which for some purposes is more important than a high pulling force, for which I set up no criterion. From the standpoint of constancy, the ratio H_c to I_r may prove useful. For example, in soft iron the ratio is 0·0021 and in hardened iron 0·0115—5·5 times as great; the products of H_c and I_r are 1,710 and 1,840 respectively. Here the products are not very different, and it is the ratio which shows that hardened iron is more suitable for a *stable* magnet than soft iron. Again, I have shown that chilled cast-iron produces very constant magnets, and I find from Prof. Thompson's table that its ratio of H_c to I_r is nearly the highest in the list. Even terra-cotta, which Prof. Thompson says may have a ratio exceeding that of any known brand of steel, has been found to preserve its magnetism for centuries, and is thus favourable to the view that a high ratio is indicative of high permanence. In short, the product $H_c \times I_r$ and the ratio $H_c \div I_r$ are not rival expressions for the same thing, but represent different quantities, and it is a mistake to confuse them. The product gives information regarding the magnitude of the hysteresis loop, the ratio

deals with its shape. The ratio measures the difficulty with which magnetism is removed; it indicates the general slope of the demagnetisation curve, and is important when treating of the loss of magnetism as susceptibility is when treating of the growth of magnetism. I need hardly say I did not know that Mr. Matthews had very recently drawn attention to the use of the ratio H_c to I_r, and had proposed to call it "permanence." The coincidence with my suggestion is purely fortuitous, because, as my letter shows, the subject had been before me when carrying out experiments some years ago.

Prof. THOMPSON replied as follows: My sole point of view was the suitability of steel for making permanent magnets, for which purpose it is irrelevant to consider either soft iron or hard iron. I have pointed out in my lecture of 1912 that a certain tool steel having a coercive force of 85, even though it have a remanence of only 370, is better for short bar magnets than any other tungsten steel; and, indeed, it is better for that purpose than any known kind of steel whatever, even though some have a remanence of 800 or more. It is of no value whatever for the permanence of a magnet's magnetism, whether measured by tractive power or by magnetic moment, that, with a given (high) coercive force, *the remanence should be low*; but that misleading proposition is the logical meaning of the proposal to take as the criterion of permanence the *ratio* of coercive force to remanence. Stability depends essentially on the highness of the former—on its absolute magnitude, not on its magnitude relating to the latter.

XIV. *A Galvanic Cell which Reverses its Polarity when Illuminated.* *By* Alan A. Campbell Swinton.

Received December 31, 1914.

In ordinary cells such as are used to show the alteration of the resistance in selenium under the action of light, the selenium is held between two metallic wires or strips, the light penetrates into the selenium at right angles to the direction of the electric current, and an exterior source of E.M.F. is employed.

The writer has constructed several selenium cells in which the selenium instead of being enclosed between two solids, as mentioned above, was contained between a solid and an electrolyte, when, as the latter was transparent, the light could be caused to pass through the selenium in the same or in the opposite direction to the electric current.

The selenium was spread upon a piece of copper which had previously been tinned to make the selenium more adherent, and then cooked for several hours in the usual manner, the uncovered portions of the copper being varnished with enamel, so that the electrolyte was not in contact with any part of the copper, but only with the selenium. To this end it was found important that there should be no cracks or pores in the selenium coating.

The other electrode consisted of a metal or carbon plate having a hole cut in it the same size as the selenium covering on the first plate, and the two plates were mounted parallel to one another in a coil with a glass window, so that light from the arc in a lantern could be projected through the window and the aperture in the metal or carbon plate on to the selenium surface.

Plain tap water was used as the electrolyte, any acid or alkali being apt to cause the selenium to come away from the copper. As in any case the electrical resistance of the selenium is very great, the extra resistance of the tap water was found of small account. It was found that such cells could be used in the ordinary way to show what is usually called the variation of the resistance of the selenium in light and in darkness with current derived from a separate battery. Some of the cells thus constructed gave good results, though not better than others constructed in the ordinary way.

If the perforated plate was made of zinc, the latter being electro-positive to selenium, the cell then worked as its own battery, the result of light falling on the selenium being to cause a larger flow of current. This might either be because the light caused the selenium to become more electro-negative, or because it reduced the resistance of the selenium. Subsequent tests, in which a Dolezalek quadrant electrometer was used instead of the galvanometer that had previously been employed, showed that the light caused an increase in the E.M.F.

If, however, instead of zinc, carbon or copper was employed for the perforated plate, we had the interesting result that while the selenium proved electro-positive to the carbon or copper in the dark, it immediately became electro-negative to carbon or copper the moment it was illuminated. The reversal of the polarity of a cell of this kind was easily shown on a D'Arsonval moving-coil galvanometer of some 285 ohms resistance, large and about equal deflections on the two sides of zero being observed in contrary directions when the light was turned on and off. There was rapid polarisation in either direction, but this polarisation was instantaneously destroyed by reversal of polarity by changing from light to darkness, or vice versa. Measurements were also made with the Dolezalek electrometer, and the E.M.F. proved to be from 0·06 to 0·11 volt in each direction. Tests made with a spectrum showed that the orange and red had the most effect, the violet appearing to have no effect at all. The results differ from those obtained by the late Prof. Minchin with his photo-electric cells of selenium and aluminium in methyl-alcohol and acetone ("Phil. Mag.," 5-31, pp. 229-234, 1891), inasmuch as these latter gave no current at all unless illuminated. Again, as in the writer's experiments both electrodes had approximately an equal amount of surface illuminated, the results seem to have no connection with those studied by Becquerel, who found that in an electrolytic cell a difference of potential can be obtained when one electrode is in darkness and the other is illuminated. Moreover, the effect with the selenium was found to be hundreds of times more powerful than what can be obtained by merely illuminating one of two similar electrodes and keeping the other in darkness.

The result with the selenium would appear to be consonant with the action of the light being to cause the emission from the selenium into the electrolyte of negative corpuscles. In

any case, the results point to the so-called alteration in the resistance of selenium under illumination being in reality due to an E.M.F. being generated.

The writer is much indebted to Mr. A. L. Davis for his skill and patience in constructing the cells and for his assistance in carrying out the investigations.

ABSTRACT.

If two plates—one of zinc and the other of tinned copper coated on one surface with selenium and varnished with enamel over the remainder of its surface—are immersed in tap-water, the electric current through a galvanometer connected to the plates shows that in the dark the zinc is electro-positive to the selenium, while the result of light falling on the selenium is to increase the effect.

If, however, instead of zinc, carbon or copper is employed for the non-coated plate, the interesting result is obtained that, while the selenium proves to be electro-positive to the carbon or copper in the dark, it immediately becomes electro-negative to carbon or copper the moment it is illuminated, this being easily shown by the deflections of the galvanometer in contrary directions as the light is turned on and off.

DISCUSSION.

Prof. T. MATHER asked if the increase of current produced by the illumination was proportional to the E.M.F. which the author had detected with the quadrant electrometer.

Mr. S. D. CHALMERS asked if heat waves would produce the effect.

Mr. W. DUDDELL commented on the fact that the visible rays were most effective while the ultra-violet rays were ineffective, and contrasted the result with the recent discovery that selenium is very sensitive to X-rays.

Mr. D. OWEN asked if the action was instantaneous. If so, by using intermittent illumination one should get a strong effect produced in a telephone circuit.

Mr. G. D. WEST thought that the ineffectiveness of the ultra-violet radiation might be due to the absorption of these rays which was bound to occur in the electrolyte.

The AUTHOR, in reply, said that the extreme instability of the cell made it impossible to co-ordinate readings of the current and the E.M.F. produced by illumination. He did not think heat waves were able to produce the effort, as passing the light through a solution of alum did not greatly reduce it. The action, he thought, was practically instantaneous. He agreed that the absorption of the cell would probably account for the ineffectiveness of the ultra-violet rays.

XV. *An Investigation of the Photographic Effect of Recoil Atoms.*

By A. B. Wood, *M.Sc., and* A. I. Steven, *M.A., B.Sc., University of Liverpool.*

Received January 13, 1915.

Introduction.

Measurements of the ranges of α-particles by different methods have shown that the ionising, phosphorescent and photographic actions cease after the α-particles have traversed the same distance in air, although they still possess nearly 40 per cent. of their initial energy. It seems reasonable to suppose, therefore, that these three properties of the α-rays must be ascribed to the same cause.

Work of Wertenstein* and one of us† has shown that the ionisation produced by a recoil atom is about 10 times greater over its range than that produced by the corresponding α-particle over the same part of the range.

It thus appears reasonable to expect that recoil atoms will produce phosphorescence and will affect a photographic plate. The object of experiments to be described was to investigate any photographic action if it occurs.

In the majority of cases the recoil atom is radio-active (*e.g.*, recoil atoms from RaA, RaC, ThC, Act.C), hence any photographic action due to energy of recoil is masked by the subsequent disintegration of the recoiling atom. Polonium is the ideal substance for the purpose of the experiments, since the atom recoiling from it, when it expels an α-particle, is itself rayless, and consequently any effect produced by it must be due entirely to the energy of its recoil.

Experimental Procedure.

There still remains the separation of the effects due to the α-particles and those due to the recoil atoms. Two methods have been adopted :—

(A) By "absorbing" the recoil atoms.

(B) By deflecting the α-particles in a magnetic field.

* Wertenstein, "Comptes Rendus," 152, p. 1657, 1911.

† Wood, "Phil. Mag.," October, 1913.

(A.) *Absorption Method.*

In this method a very active preparation of polonium, deposited on a polished platinum surface, was placed at the end of a brass "light-tight" vessel, which could be evacuated to any desired pressure. The rays emitted from this source, after passing through two parallel slits 1 mm. wide and 1 cm. apart, fell on a sensitive photographic plate, one half of which was covered with tinfoil of thickness sufficient to prevent the α-rays from striking the film beneath it. The experiment now consisted of two distinct parts: (a) The pressure in the chamber was adjusted so that the recoil atoms were just prevented from reaching the plate. This pressure, a few centimetres, could easily be calculated from a knowledge of the range of recoil atoms at atmospheric pressure and the distance apart of the polonium source and the photographic plate. By means of a shutter arrangement an exposure of any required period could be given.

(b) The pressure was now reduced to about 0·001 mm. by means of a Gaede pump, until practically all the recoil atoms could reach the plate, the α-rays, of course, reaching it as before. Now, the other half of the plate was exposed for the same length of time, the part previously exposed being covered with tinfoil.

In this way one half of the plate was affected by α-particles only, whilst the other half was affected by both α-particles and recoil atoms. On developing the plate a narrow continuous band was observed across it, one half being produced in experiment (a), the other in (b). In nearly every case, that half of the band produced in experiment (b) was slightly darker than that produced in experiment (a).

It might appear from this that the effect of the α-particles and recoil atoms together was greater than the effect produced by the α-particles alone, thus showing that the recoil atoms have a photographic action. It must be considered, however, that in experiment (a), when the pressure in the chamber is several centimetres of mercury, fewer α-particles reach the plate on account of scattering than in experiment (b), when the pressure is extremely low. Consequently, the slight difference of intensity observed might be due to this absorption or scattering effect alone. Hence it became necessary to adopt some other method by which this effect could be eliminated.

(B.) *Magnetic Deflection Method.*

Makower and Evans* have shown in the case of RaB recoiling from RaA that the deflection of recoil atoms in a magnetic field is only half that of the corresponding α-particles. Assuming that this is the case for other substances emitting α-particles, we have here a convenient method of separating the two effects. The apparatus used is essentially similar to that employed by Makower and Evans† in the determination of "e/m" and "v" for the atoms recoiling from RaA, the only difference being that in our experiments the recoil atoms were received directly by a photographic plate. A powerful electromagnet, giving a field of about 10,000 gauss with pole-faces 3·5 cm. × 1 cm. and a gap 1 cm., was employed to deflect the beam of rays issuing through the slits. Equal exposures were made with the field first in one direction, then reversed, the chamber being evacuated to a very low pressure by means of a Gaede pump.

If both α-particles and recoil atoms affect a photographic plate, then we should expect to find, on developing the plate, four lines, the two outer ones being due to α-particles and the two inner ones being the result of the impact of recoil atoms. Even with the longest exposures possible, limited by the heating of the magnet, only two lines about 1mm. wide and 0·4 cm. apart (obviously due to α-particles) were observed.

Now, Wertenstein‡ and one of us§ have shown that the penetrating power of recoil atoms is only about $\frac{1}{500}$th that of the α-particles, consequently the negative result just mentioned might be explained on these grounds—*i.e.*, the recoil atoms do not penetrate sufficiently deep into the film to produce an observable effect. To get over this difficulty, instead of using ordinary gelatine-coated plates, Schumann plates (a thin film of silver chloride on glass) were employed; the recoil atoms had thus a greater advantage in affecting the sensitive layer. Many experiments were made using these plates, but the results were either entirely negative or so uncertain as to be quite useless.

* Makower and Evans, "Phil. Mag.," Vol. CXIX., November, 1910, p. 882.
† Makower and Evans, *loc. cit.*
‡ Wertenstein, *loc. cit.*
§ Wood, *loc. cit.*

Summary and Discussion of Results.

An attempt has been made by two distinct methods to demonstrate the photographic action of recoil atoms from polonium. The method of separating the α-particles from the recoil atoms by means of a magnetic field is undoubtedly the more conclusive of the two methods employed in this investigation; but, with the sources of polonium at our disposal, the results show that the photographic effect, if any, of the recoil atoms is too small to be observed. This negative result may possibly be due to either or both of two causes:—

1. The activity of the polonium source used was not sufficiently great;

2. The recoil atoms do not penetrate sufficiently deep into the sensitive layer to produce an appreciable photographic effect.

By employing a source of very great activity both these difficulties might be overcome and a more conclusive result obtained.

It might be mentioned here that one of us has carried out a series of experiments on the possibility of recoil atoms producing scintillations on a zinc sulphide screen, but the results in these experiments also were inconclusive.

Our best thanks are due to Prof. Sir E. Rutherford for the loan of the source of polonium, and to Prof. L. R. Wilberforce for his kind interest in the experiments.

ABSTRACT.

The ionising, phosphorescent and photographic effects of the α-particles from a radio-active substance entirely cease when the particle still retains about 40 per cent. of its kinetic energy. It appears possible, therefore, that the recoil atoms from a radio-active source should be able to affect a photographic plate, for though the range of a recoil atom is only about $\frac{1}{100}$th of that of the α-particle shot off from it, the ionising effect has been shown by Wertenstein and one of the authors to be 10 times as powerful over the corresponding range as that of the α-particle.

Attempts have, therefore, been made to demonstrate this action in the case of the recoil atom from polonium, this substance being chosen on account of the inactive nature of the recoiling atom. Two distinct methods were employed: (1) The recoil atom was "absorbed"; (2) the difference of deflection of the α-particle and the recoil atom in a strong magnetic field was utilised in order to attempt to separate their effects. "Schumann" plates were used as being most easily penetrable, but in all cases the results were negative or inconclusive. This is probably due to the fact that the recoil atoms are not able to penetrate sufficiently deeply into the sensitive layer to render the grains developable.

XVI. *Magnetic "Character" Figures: Antarctic and International.* *By* C. CHREE. *Sc.D., LL.D., F.R.S.*

RECEIVED FEBRUARY 2, 1915.

§ 1. SINCE January, 1906, an international scheme has existed for assigning to individual days magnetic "character" figures "0" (quiet), "1" (moderately disturbed), "2" (highly disturbed). These figures have been assigned independently at each co-operating observatory, the observer taking into account only the curves of his own station. The results are sent in quarterly to the Director of De Bilt Observatory in the Netherlands, who undertakes the analysis and publication. At the year's end he issues a list ascribing to each day a "character" figure which represents the mean estimate of the co-operating stations, the results being given to 0·1. On these figures he bases the selection for each month of five "quiet" days, which are recommended for the deduction of the regular diurnal variation for international purposes. A "day" in this connection means a period of 24 hours commencing at Greenwich midnight.

For purposes of my own I have utilised the international data for the selection of the five most disturbed days of each month, regarded as the days having the largest "character" figures. Occasionally the question arises which of two days having equal "character" figures should occupy the fifth place. In such a case preference has been given to a day immediately adjacent to a highly disturbed day.

The impression produced on the mind by a magnetic trace depends a good deal on the sensitiveness of the magnetograph. With very high sensitiveness, such as 1 mm. $= 1\gamma (1 \times 10^{-5}$ C.G.S.) disturbances catch the eye which on an insensitive record, such as 1 mm. $= 10\gamma$, seem insignificant. Some stations, again, are much more disturbed than others, while different observers have different ideals. Thus the standard differs at different observatories. What one observer accepts at once as a "2" day, another equally readily accepts as a "1". Further, as I have shown elsewhere,* one's estimate of disturbance is apt to be influenced by the presence or absence of large disturbances about the time. If one maintained an absolutely uniform standard, class "0" would be but slightly represented

* Roy. Soc., "Phil. Trans.," A. Vol. 213, p. 245.

in a disturbed season, and class "2" in a quiet season. Thus even the final international figures do not, I think, have an exact physical significance, but they do at least discriminate with very considerable accuracy between the days of the same month.

The large majority of the stations contributing to the international lists lie between 55°N. and 25°N. lat. The extreme latitudes represented are 59·7°N. (Pavlovsk) and 31·7°S. (Pilar). There are observatories further south than Pilar in British territory from which figures would be valuable.

There is a great difference magnetically between stations in moderate and high latitudes. At all stations, so far as I am aware, which have been occupied within the Arctic or Antartic circle, magnetic disturbances have shown a persistence and amplitude unparalleled in temperate or torrid zones. There are grounds for believing a connection between magnetic disturbances and auroras of a rapidly oscillating type. The frequency of aurora in Europe diminishes rapidly as one moves southwards from the auroral belt. Auroras are frequent occurrences as far south as Shetland, but very rare in southern England. According to Prof. Kr. Birkeland's* theory, which has a good deal to say for itself, auroras and magnetic storms are alike due to flights of electric ions discharged from the sun. When these ions enter the earth's magnetic field they describe spirals round the lines of force, and none which do not possess an exceptionally high velocity, closely approaching that of light, can get near the earth, except within a moderate angular distance of the magnetic poles. There may thus be an almost continuous advent of ions with consequent aurora in high latitudes, and but very occasional advent in temperate latitudes. The disturbance due to the ions present in high latitudes is, of course, not *wholly* limited to these latitudes, but only faint effects would ordinarily be experienced in remote regions. On this view there need be no close connection between the prevalence of disturbance in mid-Europe and in high latitudes, especially high southern latitudes. Unless on the relatively few occasions when numerous ions of high velocity are discharged from the sun, there are also unusually copious discharges of the slower moving ions, the presence of a magnetic storm in central and southern Europe would not necessarily

* "Expédition Norvégienne, 1899-1900." "The Norwegian Aurora Polaris Expedition, 1902-1903," Vol. I., Sects. I. & II. Also numerous Papers in the "Comptes Rendus" of the French Academy since 1908.

imply any special disturbance in the Arctic and Antarctic. Conversely, the presence of quiet conditions in mid-Europe would raise but a slight presumption of quiet conditions in high latitudes. Thus theoretical as well as practical considerations indicated the desirability of examining the records obtained during Captain Scott's last Antarctic Expedition, to see whether there is any marked connection between the magnetic "character" of days in high and moderate latitudes. It was also investigated whether the so-called 27-day period manifests itself in the Antarctic.

§ 2. In previous investigations* of the 27-day period I have employed two sets of magnetic data, viz., the absolute daily ranges of the magnetic elements and "character" figures, either the international figures published at De Bilt, or figures assigned by myself to Kew curves, for years for which no international data exist. Absolute daily ranges have been got out for the Antarctic curves, but they refer to days in the local time of 180° longitude, and the additional labour necessary to determine a second set of ranges applicable to days G.M.T. hardly seemed justified. Thus I decided to assign and employ "character" figures.

Thanks to the assiduous care of the Antarctic magnetic observers, Dr. G. C. Simpson and Mr. C. S. Wright, there are but few gaps in the record extending from the beginning of February, 1911, to near the end of November, 1912. There were, however, a few days on which the loss of trace rendered it impossible to assign a "character" figure satisfactorily. A few hours' trace may justify one in assigning a "2," but unless the day's trace is very nearly complete it is hardly safe to assign a "0."

Table I. gives the dates of the five most disturbed and the five international quiet days of each month, along with the corresponding "character" figures assigned by myself to the Antarctic curves. A few days are left blank for the reason stated above. The "character" figures are taken from a list including all days for which adequate Antarctic records existed. When making the estimate I did not consult the international list, so there was no possible prejudice. Before actually assigning the figures, I made a general inspection of all the Antarctic curves, so as to set myself a standard of disturbance likely to discriminate adequately between the days of all the

* Roy. Soc. "Trans." A. Vol. 212, p. 75; Vol. 213, p. 245; "Proc." A, Vol. 90, p. 583.

months. If I had applied such a standard as is applied to Kew curves, most of the summer months would have received no 0's and very few 1's. Even as it is, the number of 2's allotted is much greater than at the average station co-operating in the international scheme.

TABLE I.—*Antarctic "Character" Figures on International Disturbed and Quiet Days.*

Dates.											Antarctic character figures.									
1911.	Disturbed days.					Quiet days.					Disturbed days.					Quiet days.				
February ...	2	13	21	22	23	11	12	15	19	20	2	2	2	2	2	1	1	1	1	0
March	20	21	23	25	26	10	11	12	17	18	2	2	2	2	2	0	0	0	1	0
April	8	9	16	17	30	5	13	14	15	26	2	2	2	2	2	0	–	0	1	1
May	7	14	15	16	31	1	4	13	22	24	2	2	2	2	2	0	0	0	1	0
June	4	5	9	10	11	3	17	18	19	25	1	2	1	2	1	1	0	0	0	0
July	1	17	19	28	29	13	14	15	16	26	2	1	2	2	2	1	1	0	1	0
August	19	23	24	25	26	7	8	10	11	29	2	2	2	2	2	1	0	1	0	1
September .	16	19	20	21	22	2	3	14	25	26	2	1	2	2	2	1	0	1	0	1
October	9	10	11	17	18	1	5	15	23	28	2	2	2	2	2	0	0	2	1	1
November...	3	9	13	14	15	1	7	22	23	24	2	2	2	2	2	1	1	1	1	1
December...	6	11	17	26	31	2	9	21	22	23	2	2	2	2	2	2	0	2	1	2
1912.																				
January	11	12	13	17	22	2	15	16	26	27	1	2	2	2	2	1	1	1	1	0
February ...	12	13	16	17	26	5	6	15	20	21	2	1	1	2	–	0	1	1	1	0
March	7	8	9	21	29	4	17	18	19	24	1	2	1	1	1	0	0	0	0	1
April	5	6	10	15	16	1	8	11	21	28	2	2	2	2	2	0	0	0	0	0
May	5	6	12	13	14	1	16	22	23	26	2	1	2	2	2	1	0	0	0	1
June	1	8	9	10	28	5	6	15	19	20	2	2	2	2	1	0	1	0	0	0
July	3	4	5	27	31	10	11	12	15	24	1	2	2	1	1	0	0	0	0	0
August	5	6	18	19	22	4	8	12	13	26	1	1	2	2	2	0	0	0	0	1
September .	4	17	18	23	24	2	15	16	27	28	1	2	2	1	2	0	0	1	0	0
October	1	11	13	14	15	2	5	18	19	31	2	2	1	2	2	1	1	1	0	1
November...	10	11	14	16	22	3	12	21	29	30	2	2	–	2	–	1	1	–	–	–

A glance at Table I. shows that the association between Antarctic disturbances and those in moderate latitudes was very intimate. Of the 107 international highly disturbed days for which Antarctic records existed, 85 got a "2," and not a single one a "0." Only one day which had an international "character" figure as large as 1·3 failed to get a "2," and it was a day on which several hours' trace was missing, so that allotting it a "1," as I did, was rather a rash proceeding. On the other hand, 57 of the 106 international "quiet" days got a "0" and only four got a "2." Of these, three occurred in December, 1911, a month to which I allotted sixteen 2's and only one 0. Moreover, all three were essentially quiet except for a few hours near midnight, and the disturbance was so moderate that it was open to doubt whether a "1" or a "2" was fairer.

The fourth international quiet day which got a "2" was also close to the border line between "1" and "2." In only one of the four cases was there any suggestion from the international data that there had been a disturbance peculiar to high latitudes. On that occasion a "1" had been allotted at Pilar, the most southern station, and at Sitka, one of the most northern, every other station with one exception assigning a "0."

§ 3. Various other interesting conclusions could be derived from Table I., but they are more readily recognised in Table II., which analyses the data from the different months. The Antarctic selected disturbed and quiet day data for November, 1912, were derived from only three disturbed and two quiet days, and only 15 days were available altogether, so the results for that month are somewhat uncertain.

TABLE II.—*Mean Monthly and Annual "Character" Figures.*

—	1911.						1912.					
	International.			Antarctic.			International.			Antarctic.		
	Dist.	All.	Quiet.	Dist.	All.	Quiet.	Dist.	All.	Quiet.	Dist.	All.	Quiet.
January	...	...	...	...	...	...	0·98	0·42	0·02	1·8	1·26	0·8
February ...	1·46	0·89	0·36	2·0	1·54	0·8	1·02	0·49	0·12	1·5	1·00	0·6
March	1·48	0·78	0·08	2·0	1·26	0·2	1·08	0·45	0·02	1·2	0·83	0·2
April	1·48	0·76	0·14	2·0	1·28	0·5	1·12	0·45	0·08	2·0	0·73	0·0
May	1·28	0·70	0·16	2·0	1·23	0·2	1·18	0·47	0·10	1·8	0·77	0·4
June	1·12	0·53	0·04	1·4	1·00	0·2	0·98	0·47	0·08	1·8	0·86	0·2
July	1·26	0·61	0·12	1·8	1·32	0·6	1·06	0·41	0·02	1·4	0·70	0·0
August	1·30	0·53	0·10	2·0	1·06	0·6	1·12	0·49	0·02	1·6	1·03	0·2
September .	1·30	0·50	0·06	1·8	0·97	0·6	1·22	0·47	0·02	1·6	0·97	0·2
October	1·30	0·59	0·06	2·0	1·26	0·8	1·20	0·46	0·02	1·8	1·23	0·8
November...	1·42	0·49	0·04	2·0	1·47	1·0	1·06	0·45	0·00	2·0	1·53	1·0
December .	1·30	0·45	0·08	2·0	1·48	1·4	...	...	...	...	...	...
Means	1·34	0·62	0·11	1·91	1·26	0·63	1·09	0·46	0·04	1·68	0·99	0·40

As already indicated, the necessity of discriminating adequately between the different days of the same month tends to variation in the international standard, of such a nature as to minimise the difference between different years. The fall in disturbance, however, between 1911 and 1912 was so large that it is readily recognised in the international figures in Table II., whether one regards the disturbed day, the quiet day, or the all-day results. There is also, it will be seen, a marked decline in the Antarctic "character" figures. Thus in this case, at least, the fall in disturbance was common to high and

temperate latitudes. According to statistics advanced by some authorities, auroras have been more numerous at stations to the north of the Arctic aurora belt during years in which they have been less numerous at stations to the south of it. Thus it would not have been safe to assume *a priori* that disturbances near a magnetic pole and those in temperate latitudes wax and wane together.

Another phenomenon clearly visible in Table II. seems peculiar to the Antarctic. In temperate latitudes disturbance is most prominent towards the equinoxes. In the Antarctic the equinoctial months, while decidedly more disturbed than the winter months, are less disturbed than the summer months. This phenomenon was also clearly shown by the magnetic curves obtained in 1902 and 1903 by the first Scott Antarctic Expedition.

TABLE III.—*Antarctic "Character" Figures. Sums on Selected and Associated Days.*

	$n-2$	$n-1$	n	$n+1$	$n+2$	$n+25$	$n+26$	$n+27$	$n+28$	$n+29$	$n+30$
Disturbed and associated days:											
1911 (55 days)	55	76	106	98	88	63	72	83	90	85	76
1912 (49 „)	44	58	81	72	53	48	55	64	57	56	45
Total (104 days)	99	134	187	170	141	111	127	147	147	141	121
Quiet and Associated days:											
1911 (55 days)	67	54	35	46	66	65	50	47	54	67	74
1912 (49 „)	42	35	16	31	44	37	41	41	44	49	50
Total (104 days)	109	89	51	77	110	102	91	88	98	116	124
Disturbed and associated, less quiet and associated:											
1911 (55 days)	−12	+22	+71	+52	+22	− 2	+22	+36	+36	+18	+ 2
1912 (49 „)	+ 2	+23	+65	+41	+ 9	+11	+14	+23	+13	+ 7	− 5
Total (104 days)	−10	+45	+136	+93	+31	+ 9	+36	+59	+49	+25	− 3

§ 4. In order to ascertain whether the 27-day period is recognisable in the Antarctic, the "character" figures which had been assigned were entered in tables of which Table III. constitutes a summary, in columns headed $n-2$ to $n+2$ and $n+25$ to $n+30$. Here n stands for a representative selected day, whether one of the five disturbed or one of the five international quiet days of the month, $n-1$ and $n+1$ represent the days immediately preceding and succeeding, $n+25$ the day 25

days later, and so on. None of the selected days for November, 1912, appeared in column n because there were no corresponding data for columns $n+25$ to $n+30$. What appears in Table III. is the *sum* of the "character" figures, allowance being made for one or more days missing in some of the columns. For instance, in 1912, in the case of the disturbed and associated days, only 48 days' "character" figures were really available for column $n+2$. Their sum 52 was brought up to 53, as the integer nearest to $(49/48)\times 52$.

The 27-day period is clearly shown in the case alike of the disturbed and the quiet days, and still more clearly in the difference figures, obtained by substracting the totals for the quiet and associated days from the corresponding totals for the disturbed and associated days

A similar procedure was applied to days that were from 30 to 25 days earlier than the selected disturbed and quiet days. In this case the selected days of February, 1911, could not be utilised. Table IV. gives the results obtained for the columns $n-30$ to $n-25$. The results for the columns $n-2$ to $n+2$ differed so little from the analogous results in Table III. that it seemed unnecessary to give them. While Table III. suggests a period distinctly longer than 27 days, Table IV. would make the period more nearly 26 days. This is not improbably connected with the lack of symmetry which Table III. discloses in the primary pulse. The "character" sum in column $n+1$ is greater for disturbed and less for quiet days than the corresponding "character" sum in column $n-1$. Thus the centre of the primary pulse would seem to fall well after the middle of the representative selected day.

TABLE IV.—*Antarctic "Character" Figures. Sums on Associated Days.*

	$n-30$	$n-29$	$n-28$	$n-27$	$n-26$	$n-25$
Disturbed associated days:						
1911 (50 days)	49	51	67	79	84	80
1912 (55 „)	61	56	55	67	70	67
Total (105 days).....	110	107	122	146	154	147
Quiet associated days:						
1911 (50 days)	64	60	54	48	47	51
1912 (55 „)	57	51	47	49	49	47
Total (105 days).....	121	111	101	97	96	98
Disturbed associated less quiet associated (105 days).....	−11	− 4	+21	+49	+58	+49

Table V. employs the same data as Table III., but arranges them under three seasons, viz., winter (May to August), summer (November to February) and equinox (the remaining four months). The 27-day period is apparent in each season, but is less well marked in summer than in winter or equinox. This is partly explained by the smaller number of days available in summer, and partly by the fact that disturbances at that season were so large and numerous that the number of "character" 0's allotted was hardly sufficient to discriminate adequately between the days of the same month.

The sun, it should be remarked, was continuously above the horizon almost the whole of summer, and continuously below the horizon almost the whole of winter. Its being below the horizon during the latter season seems not to have militated in any way against the 27-day period.

TABLE V.—*Antarctic "Character" Figures. Sums on Selected and Associated Days.*

		$n-2$	$n-1$	n	$n+1$	$n+2$	$n+25$	$n+26$	$n+27$	$n+28$	$n+29$	$n+30$
Winter, 40 days	Dist., &c.	37	50	69	64	51	40	49	57	57	51	41
	Quiet, &c.	34	25	12	23	31	34	26	25	34	37	39
Excess of dist'bd, &c.		+ 3	+25	+57	+41	+20	+ 6	+23	+32	+23	+14	+ 2
Equinox, 40 days	Dist., &c.	36	50	72	64	57	45	48	59	59	61	53
	Quiet, &c.	42	34	16	25	42	40	39	39	42	51	53
Excess of dist'bd, &c.		− 6	+16	+56	+39	+15	+ 5	+ 9	+20	+17	+10	0
Summer, 25 days	Dist., &c.	27	35	47	44	34	27	31	32	32	30	28
	Quiet, &c.	34	31	23	30	38	29	27	25	23	29	33
Excess of dist'bd, &c.		− 7	+ 4	+24	+14	− 4	− 2	+ 4	+ 7	+ 9	+ 1	− 5

§ 5. The mean of the Antarctic "character" figures, notwithstanding the much higher standard of disturbance employed, was much in excess of the corresponding international mean. Thus it appeared desirable for purposes of comparison to express the results as percentages of some standard value. Two quantities suggested themselves as standards, 1° the mean "character" figure from all days of the year, and 2° the amplitude of the primary pulse, *i.e.*, the excess of the "character" figure of the representative disturbed day over that of the representative quiet day of the year. Both have been used. The Antarctic data referred to both standards are given in Table VI. The international data referred to the two standards appear separately in Tables VII. and VIII. These two last

tables are not confined to 1911 and 1912, but include all the years for which international data have been published.

To explain the figures, consider the Antarctic results for 1911 in Table VI. The mean "character" figure for 1911 was 1·26. There were 55 selected disturbed and 55 selected quiet days. The sum of the "character" figures for 55 average days is $55 \times 1{\cdot}26$ or 69·3. The sums for the 55 selected days were respectively: disturbed 106, quiet 35. The difference of these sums—or amplitude of the primary pulse—71, expressed as a percentage of 69·3, is to the nearest integer 102. The sums for the 55 days which followed 27 days after the selected days were respectively: associated disturbed days 83, associated quiet days 47, and again to the nearest integer $100\,(83—47)/69{\cdot}3=52$.

The figures in the second line of Table VI. were got in an exactly similar way, using as standard 0·99, the mean "character" figure for 1912.

TABLE VI.—*Antarctic "Character" Figures. Differences Disturbed and Associated Days less Quiet and Associated Days as percentages.*

		$n-2$	$n-1$	n	$n+1$	$n+2$	$n+25$	$n+26$	$n+27$	$n+28$	$n+29$	$n+30$
As percentages of mean character figure for year	1911	−17	+32	+102	+75	+32	− 3	+32	+52	+52	+26	+ 3
	1912	+ 4	+47	+134	+85	+19	+23	+29	+47	+27	+14	−10
	Mean	− 7	+40	+118	+80	+25	+10	+30	+50	+39	+20	− 4
As percentages of difference dist. less quiet on day n	1911	−17	+31	+100	+73	+31	− 3	+31	+51	+51	+25	+ 3
	1912	+ 3	+35	+100	+63	+14	+17	+21	+35	+20	+11	− 8
	Mean	− 7	+33	+100	+68	+22	+ 7	+26	+43	+35	+18	− 3

TABLE VII.—*International "Character" Figures. Differences Disturbed and Associated Days less Quiet and Associated Days as percentages of Mean "Character" Figure for Year.*

Year.	$n-2$	$n-1$	n	$n+1$	$n+2$	$n+25$	$n+26$	$n+27$	$n+28$	$n+29$	$n+30$
1906	22	92	179	96	15	6	37	47	24	7	−16
1907	17	96	179	83	2	8	31	31	17	18	29
1908	14	94	175	98	17	27	60	57	44	15	− 2
1909	29	94	195	103	25	−15	0	37	41	35	17
1910	21	83	161	80	18	5	37	56	47	22	6
1911	10	86	189	93	16	26	65	89	69	21	2
1912	0	98	233	109	−18	17	48	69	18	−14	−16
1913	52	131	228	118	40	59	88	98	62	46	34
Mean	20·6	96·7	192·5	97·5	14·5	16·7	45·7	60·4	40·3	18·7	6·8

TABLE VIII.—*International "Character" Figures. Differences Disturbed and Associated Days less Quiet and Associated Days as percentages of Difference on Day n.*

Year.	$n-2$	$n-1$	n	$n+1$	$n+2$	$n+25$	$n+26$	$n+27$	$n+28$	$n+29$	$n+30$
1906	12	51	100	54	9	4	20	26	14	4	− 9
1907	10	53	100	46	1	4	17	17	10	10	16
1908	8	53	100	56	10	15	34	33	25	8	− 1
1909	15	48	100	53	13	− 7	0	19	21	18	9
1910	13	52	100	49	11	3	23	34	29	13	4
1911	5	46	100	49	9	14	35	47	36	11	1
1912	0	42	100	47	− 8	7	21	30	8	− 6	− 7
1913	23	57	100	52	18	26	39	43	27	20	15
Mean	10·7	50·4	100	50·8	7·7	8·2	23·6	31·1	21·2	9·9	3·5

As regards the second set of percentages in Table VI., we have already seen that the amplitude of the primary pulse for 1911 was 106−35, or 71. In column $n+27$, for example, for 1911 we have $100(83-47)/71=51$ to the nearest integer. The percentage figures for the individual years in Tables VI., VII. and VIII. were all calculated to one figure beyond that recorded, and these more exact figures were employed in deducing the final means given in the tables.

Fig. 1 shows separately the primary and secondary pulses for disturbed and associated and for quiet and associated days, the numerical data represented being the percentages whose difference appears in the last line of Table VII.

Fig. 2 shows the primary and secondary difference pulses derived from the Antarctic data in the last line of Table VI. and from international "character" data in Table VIII. The sunspot maximum group of years 1906, 1907 and 1908 had a mean sunspot frequency of 53·8, while the sunspot minimum group 1911, 1912 and 1913 had a frequency of 3·5.

The numerals attached to the observational points in both figures denote the number of days from the crest (marked 0) of the primary pulse, time previous being measured to the left and time subsequent to the right. In Fig. 2 each crest 0 is at the same height above its zero line, and corresponding primary and secondary pulses and their common zero line are represented by a common type of line.

Tables VII. and VIII. show that even when data from a large number of stations are combined, the prominence of the 27-day period varies a good deal from year to year. Data from a single station are naturally exposed to greater uncertainties, but it seems a pretty safe inference from Table VI. and Fig. 2, that the 27-day period is as prominent in the Antarctic as elsewhere,

Either set of results in Table VI. makes the 27-day period more prominent in 1911 than in 1912, agreeing in this with the international figures. This is one of the reasons for including other years in Tables VII. and VIII. In the absence of evidence to the contrary, the lesser development of the 27-day period in 1912

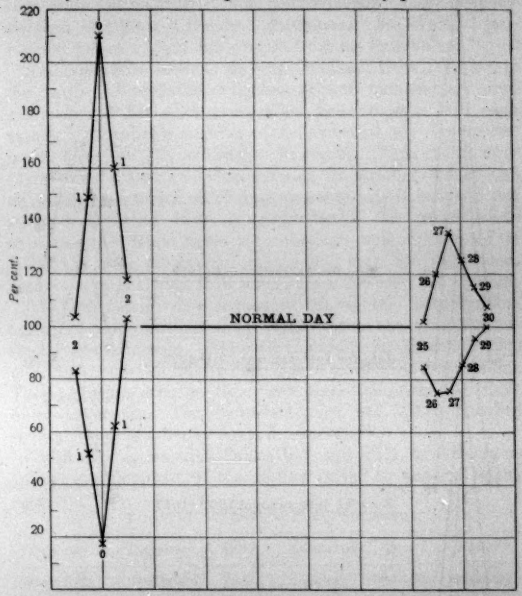


FIG. 1.—27-DAY PERIOD IN INTERNATIONAL "CHARACTER" FIGURES DISTURBED AND QUIET DAY PULSES.

might have been ascribed to the fact that that year was near sunspot minimum. On Prof. Birkeland's* theory this would seem a very natural inference. It appears, however, from Tables VII. and VIII. that the period was more developed in

* In some of his writings on the subject Prof. Birkeland seems to identify sunspots with the sources of the electrical discharges to which he ascribes magnetic storms. In others, *e.g.*, "The Norwegian Aurora Polaris Expedition, 1902-1903," Vol. I., Sect. II., p. 525, he seems to go no further than the conclusion "that sun spots and magnetic storms are both of them manifestations of the same primary cause."

1912 than in either 1907 or 1909, the former a year of sunspot maximum, while less developed than in 1913 which had even fewer sunspots. It is also more prominent in the curve for

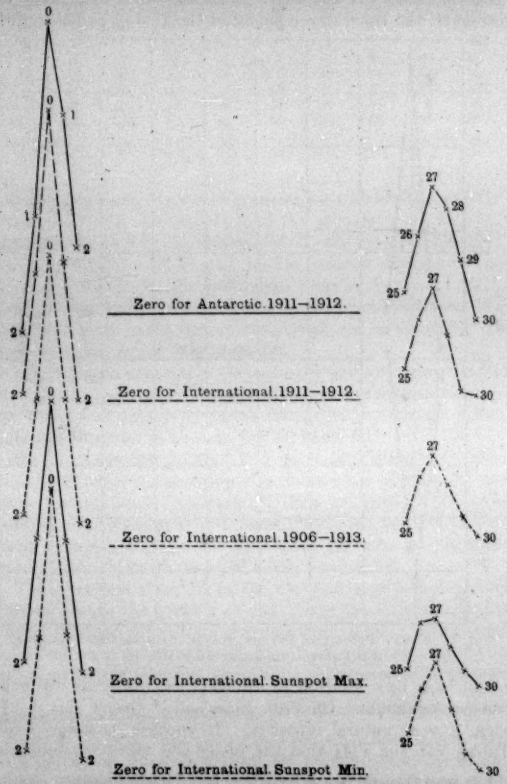

Zero for Antarctic 1911–1912.
Zero for International 1911–1912.
Zero for International 1906–1913.
Zero for International Sunspot Max.
Zero for International Sunspot Min.

FIG. 2.—THE "27-DAY" PERIOD. "CHARACTER" FIGURES (DIFFERENCES).

sunspot minimum years than in that for sunspot maximum years in Fig. 2.

§ 6. The question seemed of sufficient interest to call for a further investigation, the results of which appear in Table IX. This gives the number of occasions on which a 27-day interval intervened between selected days, disturbed and quiet. The last column gives Wolfer's sunspot frequency. The frequency assigned to 1913 is still only "provisional," but Wolfer's provisional values seldom differ much from his final values.*

Antecedents to selected days in January, 1906, fell outside the range of international figures, thus 95 months only were available. If five random days had been selected from each month the probable number of sequences of any one species would have been 78, or roughly 10 a year. Thus the number of disturbed-disturbed sequences (*i.e.*, the number of occasions on which an interval of 27 days presented itself between two selected disturbed days) is nearly double the mathematical expectation. While these sequences are well represented in 1907, the year of sunspot maximum, they are even more numerous in 1911 and 1913 when sunspots were nearly absent.

The quiet-quiet day sequences exceed the disturbed-disturbed in four years out of the eight, but on the whole are distinctly less numerous. They are especially prominent in 1906 and 1910, years neither of sunspot maximum nor minimum. Their aggregate is no less than three times that of the disturbed-quiet sequences. The disturbed-quiet and quiet-disturbed sequences are practically equal in number.

Table IX. agrees with Tables VII. and VIII. in showing no special development of the 27-day period in years of many sunspots.

TABLE IX.—*Number of 27-day Sequences.*

First day.....	Disturbed.	Quiet.	Disturbed.	Quiet.	Wolfer's sunspot frequency.
Second day...	Disturbed.	Quiet.	Quiet.	Disturbed.	
Year.					
1906	13	25	6	4	53·8
1907	19	10	9	4	62·0
1908	17	18	4	2	48·5
1909	15	17	8	11	43·9
1910	17	21	5	5	18·6
1911	25	18	5	5	5·7
1912	18	8	5	9	3·6
1913	28	15	2	3	1·2
Totals	152	132	44	43	...

* Since this was written Wolfer's final value has been published. This entails the substitution of 1·4 for 1·2 in the value for 1913 in Table IX., and of 3·6 for 3·5 in the mean value for the sunspot minimum years 1911, 1912 and 1913.

ABSTRACT.

The Paper makes use of magnetic "character" figures "0" (quiet day), "1" (moderately disturbed day), "2" (highly disturbed day) to investigate whether the incidence of disturbance at the base station of the Scott Antarctic Expedition, 1911-1912, did or did not accord with the incidence of disturbance in temperate latitudes; also whether the "27-day period" could be recognised in the Antarctic data.

A very complete set of magnetic curves was obtained by the physical observers of the Antarctic Expedition, Dr. G. C. Simpson and Mr. C. S. Wright, extending from February, 1911, to November, 1912. "Character" figures were assigned to each day's records by the author, and a comparison was made with the corresponding international figures published annually at De Bilt, Netherlands.

The incidence of disturbance in the Antarctic was found to agree closely with that shown by the international lists, in spite of the fact that the disturbances in the Antarctic were much larger and more persistent than at any of the stations co-operating in the international scheme.

The "27-day period" was clearly visible in the Antarctic records both in summer and winter, being as well developed there as elsewhere.

DISCUSSION.

Mr. DUDDELL thought the Paper very interesting. It was only by patient investigation of this kind that we should get any nearer a comprehensive theory of terrestrial magnetism.

Mr. F. E. SMITH remarked that it was described as "clearly visible" from Table II. that the disturbances were more pronounced in temperate latitudes during the equinoctial periods than in the intervening months. This was hardly obvious to the casual observer.

Prof. J. W. NICHOLSON said that Dr. Chree's Paper made the position of Birkeland's theory more curious than ever. Leaving out of consideration, as indisputable, the mathematical difficulties of the theory which Dr. Chree had referred to, he would like to indicate one point in which the theory had some support. If we were receiving particles from the sun, some might come from the corona, and might show themselves in the coronal spectrum and, later, in that of the aurora. There was, in fact, a strong correspondence between these spectra in some important particulars, and it does seem probable that we receive such particles. It is possible that the corona actually consists of the particles which are being shot from the sun and which we receive. But the spectra of sunspots are not in accordance with the view that we receive an unusual number from the spots.

Prof. O. W. RICHARDSON thought that in the case of particles streaming through the corona the auroral spectrum would be that of the atmosphere, and not that of the exciting particles. Only electrons should really be effective in reaching the earth under the proper conditions for the aurora.

Prof. NICHOLSON said that the correspondence in spectra would only be expected if the emitted particles were the corona itself, and not merely something passing through it. We do seem, as far as the evidence is trustworthy—the auroral spectrum being somewhat unreliable—to be receiving particles heavier than electrons, whether these are really responsible for the aurora or not.

Dr. A. RUSSELL said that in the Antarctic the disturbances seemed

less violent in winter than in summer, and it would be of interest to know if a similar relation held in the Arctic.

Dr. C. CHREE, in reply, said that it should be remembered that in the Antarctic the midsummer months were December, January and February. The values for these months stand out more highly. Birkeland is not definite on the question as to whether sunspots are the origin of the ions or not. One reason for associating them with sunspots is that in sunspot minima the magnetic disturbances are not pronounced, but what one fails to find is an equivalent excess of disturbance during sunspot maxima.

XVII. *The Electrification of Surfaces as Affected by Heat.* By P. E. Shaw, *D.Sc.*

Received January 20, 1915.

Introductory.

The uncertainty, as to sign, of the charge produced on unlike solids when rubbed together has long been known. Thus, it is possible to select three glass rods of different materials, A, B, C. A rubbed with B becomes +, but when rubbed with C becomes —. Again, most kinds of glass are — to flannel, but some are +. Such different action is attributable to different surface-hardness and to composition. Faraday knew of such anomalies and mentioned some in his researches.

The particular irregularity with which this Paper deals is that produced if the surfaces about to be rubbed are heated. Thus, when smooth glass is rubbed with silk we have, as shown by the action of the gold-leaf electroscope, glass+/silk —. Call this the *normal* action.

If the glass be passed to and fro in a flame for a few seconds we find on rubbing glass —/silk +. Call this the *abnormal* action. After finding this effect, I searched in several text-books and treatises (including French and German), but found mention of the phenomenon in one book only—viz., Hadley's "Magnetism and Electricity," p. 119 (MacMillan). The suggestion is thrown out there that it may be "due to the removal by the flame of the film of air condensed on the surface of the glass." There seems no definite reason for adopting this theory of the action, and the following experiments appear to disprove it. It is also mentioned in the above passage that the anomalous effect may be removed by allowing the glass to cool and then warming it again in a sand oven. I have tried this method of obliteration, and have found it fail both when the glass is resting on the hot sand and when it is buried in the sand. See also Experiment (9) below. But it may answer for one kind of glass and not for others. Two methods I have found unfailing to bring the glass back to normal: (1) Long continued rubbing with silk or cotton; (2) passing the rod of glass through the hand or a sheet of indiarubber. This is much quicker in action than (1).

If a systematic examination of the common hard solids and

rubbing materials be made the abnormal effect will be found universal, though in varying degree.

The electrification series may be written :—

+			
(Vitreous silica)	(Vulcanised fibre)	Metals	Sealing wax
Catskin	(Mica)	(Slate)	Resin
Flannel	Cotton	(Brown paper)	Sulphur
Ivory	Silk	(Gas carbon)	Gutta-percha
Rock-crystal	The hand	(Ebonite)	(Celluloid)
Glass	Wood	Indiarubber	—

The general order is taken from Ganot's "Physics," but the materials inserted in brackets are my addition. This is the normal series, but it must be understood that varying hardness and composition will cause changes in the order.

Experiments.

Commence with the pair *glass/silk*, and be careful to use throughout the same specimen of each :—

1. Having obtained the normal relation glass +/silk —, place the former in (*a*) a clear bunsen flame. We then find glass —/silk +. Render the glass normal (see above) and repeat the experience for (*b*) a smoky bunsen flame, (*c*) an alcohol flame, (*d*) a benzene flame. Since all the flames operate equally well, the effect cannot be attributed to chemical peculiarity, such as, for instance, a trace of sulphur in (*a*) or unoxidised carbon in (*b*). The effect increases up to a limit with time exposure in the flame.

2. After the glass has been in the flame there is a sticky feeling about it which may be due to moisture, but does not seem like it. The abnormal effect is greater when the stickiness is greater. But nothing is visible on the surface of the glass under a high power microscope.

3. In one instance the abnormal glass was put aside for 12 days. It remained abnormal at the end of the time.

4. Dip the abnormal surface in water and let the latter dry off. The abnormal state remains.

5. Instead of letting the water evaporate off the rod wipe it off with a cotton duster. If the duster be dry we have glass +/cotton—, but if damp we have glass —/cotton +. The latter effect will pass away in time, as friction continues, leaving the rod normal.

6. After abnormal glass has been excited for some time by silk we get glass +/silk —. Now remove the charge from the glass (*over*, not *in*, a flame) and again excite with silk ; we often

find glass —/silk +. This after effect is weak and is soon removed.

7. Prepare a small electric furnace. A test tube of vitreous silica is surrounded by a heating coil and the whole packed in slag wool. Now place a rod of normal glass in the furnace by the side of a platinum thermometer. Raise the temperature to 650 deg., which is just short of the melting point of the glass used. Remove the glass and allow it to cool quickly, or, leaving it in the furnace, shut off the current and let it cool slowly. In either case the glass becomes abnormal when cold. Next raise the glass to 720 deg. and melt it. The result is as before.

8. Raise the furnace to 800 deg. Place the normal glass rod in it and remove after two seconds. When cold the glass is abnormal to silk.

9. Ten seconds exposure of the glass rod in the furnace will discharge it whether it is normal and charged + or abnormal and charged —. But it always emerges abnormal.

10. Exposure of the glass in the blowpipe flame for one second renders it as abnormal as 20 seconds in the furnace. In the first case the surface is barely warm, in the second it is very hot.

11. Place normal or abnormal glass in a blowpipe flame and melt the glass and allow to cool quickly. When cool it is always abnormal.

12. Place normal glass in the cold furnace. Raise the temperature slowly to 700 deg. Then stop the current and let whole cool very slowly, thus annealing the glass. When cold the glass is abnormal. But the effect is so slight that we here have confirmation of the theory that the abnormal effect is due to strain.

13. Excite normal glass with indiarubber. It becomes +. Pass it through a flame and excite again. It is feebly —. Excite with silk it is strongly —. Thus glass, which normally is 5th in the series, comes when abnormal below indiarubber, which is 15th in the series.

The foregoing experiments show that (*a*) glass when heated has its surface so transformed that it descends in the series from its usual high place to one at least below indiarubber. (*b*) The effect is superficial, being found after fusion of the material as a whole and also after the glass is warmed slightly in a flame; though, no doubt, in the latter case, the surface layers are subject to high temperature and great strain. (*c*) The effect does not pass away by immersion in water or if the

surface is breathed on, or (*d*) by contact with the air for a week or two.

Next proceed to test solids other than glass.

Vitreous Silica/Silk.

1. Pass a normal silica rod through the flame. It is now abnormal. It holds its state while it is rubbed with silk 15 times. If glass be treated in the same way it requires only five rubs to make it normal. Thus silica shows the effect better and retains it better than glass.

2. Raise silica to the highest attainable temperature, say 1,200 deg., in the blowpipe flame, or bring it to 900 deg. in the special furnace. Whether cooled slowly or quickly it is abnormal in each case when cool.

3. The after effect observed in glass (see 6) is more pronounced in silica. This after effect is, perhaps, due to some molecules on the surface acting normally and some abnormally at the same time. The explanation of the after effect would then be as follows:—

When the abnormal glass surface is rubbed sufficiently a majority of the surface particles become normal in action, but these would at first be in a transition state. They would be unstable, readily becoming abnormal. The + charge produced on the glass is removed by the free ions above the flame, but the slight heat experienced by the surface layers when over the flame may suffice to bring these unstable particles back to abnormality. This unstable effect would then be analogous to that sometimes found in a group of many neighbouring equispaced magnets (as in Sir J. A. Ewing's models) when influenced by a transitory field.

In all the following pairs the substance placed first becomes abnormal after passing through the flame:—

Ivory/silk.
Wood/indiarubber.
Copper/indiarubber.
Copper/brown paper.
Steel/indiarubber.
Lead/indiarubber.
Brown paper/sealing wax.
Gas carbon/indiarubber.
Slate/indiarubber.
Vulcanised fibre/silk.
Ebonite/indiarubber.

Of some 18 pairs tested only one fails to give the abnormal effect. This is indiarubber/sealing wax. But rubber is the softest substance used, and the general law is that hard substances show the effect well and soft ones ill. Probably the effect occurs in rubber, but can only be observed by more delicate means.

All substances when abnormal stand below indiarubber in the electrification series.

In making these observations some general precautions are to be observed: (*a*) Many of the substances conduct more or less well. These must be mounted for a short length in a tube of glass or silica to avoid conduction and induction of the hand. (*b*) When the abnormal effect is slight, care is required to observe whether a downward movement of the gold leaf is due to charge on the rubbed body or is merely an induction effect by it. When in doubt the leaf is used discharged. (*c*) In many cases the charges are only producible by *lightly* brushing one solid by the other. Thus the — charge, when abnormal metal is excited by rubber, can only be produced in this way.

Review.

As to the cause of the abnormal effect. It cannot be attributed to any organic substance deposited by a flame, since the effect is producible without flame. It cannot be due to the removal of a layer of condensed water vapour, since after immersion in water the effect remains on the surface. Nor is it due to the deposition by flame of water on the cold surface, since flame is not necessary. Again, it seems unlikely to be caused by removal by heat of the air film, since it remains on the surface 12 days or more after the time of production, the solid being in the air during the whole time. If, then, the effect is not due to a layer *on* the surface of the solid, we must regard the surface layers of the solid as the seat of the action. Whether in producing the effect the solid be surrounded by air in a furnace or by flame, there must be sudden great agitation and subsequent strain imposed on the surface layers of molecules. Under these circumstances the surface readily produces — when excited. The process of rubbing, especially in the case of soft bodies, would relieve the state of strain and the surface would be restored to normal. Any closer conjecture would seem out of place at present.

One or two tests were applied as to the surface action. (*a*) The orientation of surface atoms in the magnetisation of steel

might have some relation to the strain effect we are considering. A strong magnetic field was applied to a steel bar both when normal and abnormal. The bar was also subject to repeated reversals of field. But no change seemed to occur in any case. (*b*) It was thought that sudden lowering (like sudden raising) of temperature might have an effect on the surface layers, causing the body to rise or fall in the electrification series. The lowest available temperature was that of freshly-prepared liquid air. This seemed to have no influence on silica, glass, brass or sealing wax, whether these were normal or abnormal. But this lowering of 200 deg. is small *cf* with the rise, 1,600 deg. in the case of a bunsen flame, so the negative result here may merely indicate that the shock is not severe enough. One objection to cooling the solid is that water vapour soon condenses from the air on the surface, and this freezes and thus alters the conditions for rubbing.

There are few recent researches directly bearing on the point raised in this Paper. Two may be mentioned :—

W. Jamieson (" Nature," 83, p. 189, April 14, 1910) found that the convex side of a bent strip of celluloid or mica is + to the hand, whereas the concave side is —. I have not succeeded in reproducing this effect. If it exists, we see that the surface under tension gives +, whereas that under compression gives —.

Sir J. J. Thomson (Camb. Phil. Soc. " Proc.," 14, p. 105, March 6, 1907) found that salts heated to 300°C. give off charges. Some salts produce +, others —. The sign of the charge remaining in the salt is always the same as that produced by friction. The salts appear to be covered by a double layer of electrification, and it is suggested that a double layer occurs on the surface of all solids. When electrification by friction takes place one or both of these layers are rubbed away.

The subject of frictional electrification has so far been handled qualitatively only; hence our present ignorance on the subject. But there should be a fruitful field of research in treating it quantitatively. Something useful might be discovered by frictional work in vacuo.

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A glass rod was raised in temperature slowly to dull redness and slowly cooled (say in two hours) to ordinary temperature. It was then found to be abnormal. The annealing here attempted may be imperfect, but if the best possible annealing

leaves the substance "abnormal" we shall have to reverse the terms normal and abnormal as used above, since the annealed state must surely be the normal one and the state attained after rubbing would be abnormal or strained.

With a view to removing the surface layers, hydrofluoric acid was applied to an abnormal glass rod. After washing in water and wiping, the glass surface was slightly abnormal. From this it might seem that the layers of glass under the surface are always abnormal. This we should expect from the experiment immediately preceding. But the result appears nugatory for two reasons: (1) A foreign substance which acts chemically on the surface upsets all the conditions; (2) the acid etches and roughens the glass surface, and it is well known that "rough" glass is in general negative to silk.

As a further test on the after effect mentioned above, a glass rod was made abnormal, then thoroughly normal—*i.e.*, it was not possible to get any after effect in the usual way. It was then put aside for several days. It was then found to be normal still. Hence we see that the normal and abnormal states are both stable if well established, but that there is an uncertain intermediate state.

As one result of these experiments we obtain a rule for discharging charged surfaces:—

Place the charged surface some distance, say 15 cm., above or at the side of a flame, *never in* the flame, and keep it there for a few seconds only at a time. If the body be kept in the flame for a second or two, or be kept over the flame for a minute or two, it will be discharged, but it will also be abnormal.

ABSTRACT.

I. The Paper deals with the anomalous electrical behaviour of various substances when subjected to heat. For example, a glass rod rubbed with silk is normally left positively electrified, but if the rod be passed through a bunsen flame, or heated in an electric furnace, and then allowed to cool, it will be found on again rubbing with the silk that the glass becomes negatively electrified. The reversal in sign, here called abnormal, can be produced (*a*) in a clear or smoky bunsen flame, (*b*) in a blowpipe flame, (*c*) in a benzene flame or an alcohol flame, (*d*) over any flame if enough time is allowed, (*e*) in an electric furnace, where heat reaches the surface affected by radiation through air.

II. Discharge of any charge + or − which may be on the surface in every one of the above cases precedes the production of the abnormal state.

III. The abnormal state can be removed from the surface by continued rubbing with silk, cotton, &c., or, better, by rubbing the surface

with the hand. It is not removed by melting (in the case of glass) or by annealing, or by the action of water or by lapse of time.

IV. It is a surface effect only.

V. An unstable intermediate condition is found after the surface has been rubbed for some time. In this condition the surface may act normally when rubbed in one way, abnormally when rubbed otherwise. This state does not last.

VI. All solids which will stand the action of a flame for a second or two appear to act, as does glass, in the above ways. The action is seen best in vitreous silica and less well in wood, metals, slate, paper, ebonite, &c. In all cases the substance rubbed is, when normal, above india-rubber the ordinary frictional list, but below it when abnormal.

DISCUSSION.

Prof. RICHARDSON said he had thought the effect might have something to do with the emission of ions when the rod was heated, but the details did not fit in with this. It probably resulted from some mechanical or chemical change in the surface molecules.

Mr. F. E. SMITH said it was a familiar fact to teachers of physics that glass would sometimes behave in unexpected ways in frictional experiments. It had always been his habit before rubbing the glass to pass it through a flame, but the glass was usually positive after rubbing with silk which had been treated with amalgam. Possibly the amalgam was responsible for the absence of any abnormal effect. It seemed significant that the abnormal effect could be destroyed by passing through the hand. The moisture of the hand was alkaline, as was also the moisture usually found condensed on substances, and which the flame would naturally remove. He suggested pouring mercury through funnels of different materials as a means of exciting electrification. This would eliminate uncertain effects in one member of the pairs and might simplify the investigations.

Dr. C. CHREE asked if the whole of the surface was found to be in one state either normal or abnormal at the same time.

Prof. HOWE asked whether, if a flame were played against one side of a glass plate, the plate would act normally on one side and abnormally on the other.

Prof. S. W. J. SMITH thought that devitrification of the glass or silica on heating might have something to do with the phenomenon, though the effect of an action of this kind would not, of course, be destroyed by the simple expedients found to be effective by the author.

Mr. G. L. ADDENBROKE mentioned some experiments of his, in which the effect of moisture on leakage of condensers had to be investigated. Usually the surface of glass was comparatively conducting, and was made much more so by warming to about 30°C. If, however, it were dried by heating, then, so long as the glass was kept slightly warmer than the air—1°C. was enough—condensation seemed to be arrested and the glass remained dry. The glass should invariably be washed with distilled water, or the condition of its surface was quite uncertain.

Dr. SHAW, in reply, said that Mr. Smith's suggestion of running mercury over the surface under test had not been tried by him. It should prove easy and useful. As to Mr. Smith's suggestion that the alkaline surface of glass and the acidic surface of silica might influence the effect, he would group that with Dr. Smith's idea of devitrification as possibly having some force in the cases of glass and silica, but not for the vast variety of other solids for which the effect has been found. Mr. Addenbrooke's remark as to thoroughly washing the glass would also seem to bear on some only of the substances used; but one would expect that in any case the normal state of the surface

as to covering layers would return in a week, which is not so. Dr. Chree had considered the possibility that the surface might be normal or abnormal in patches, but there was no evidence at hand. If the effect was attributable to strain it might be possible to get one effect on the convex side and the contrary on the concave side of a bent glass plate. Dr. Shaw had tried this effect, but had failed to observe any difference, though he was aware tension and compression were supposed to produce contrary effects. The subject raised in this Paper raised great theoretical possibilities, and should next be taken up quantitatively and the surfaces treated in vacuo.

XVIII. *Electromagnetic Inertia and Atomic Weight.* *By* J. W. Nicholson, *M.A., D.Sc., Professor of Mathematics in the University of London.*

It is generally believed that the experiments of Kaufmann and Bucherer have finally demonstrated that the mass of a swift β particle, and, therefore, of any electron, is purely of electromagnetic origin, and in accordance with the formula of Lorentz. This mass is accordingly, on the supposition that a slowly-moving electron is spherical, of the form ae^2/ac^2, where a is a numerical factor, quoted usually as $\frac{2}{3}$, and e and a are the charge and radius of the electron, c being the velocity of light in free aether. Modern theories of the nature of positive electricity in an atom—theories to which we are led inevitably by many converging lines of experiment—demand also a discrete structure for positive electricity. It must exist in portions equal to e or multiples of e, and of a size not greater than that of the electron, but of much greater mass. The similarity of these portions to electrons suggests that their mass is also purely electromagnetic, and if so, it should be subject to the same formula, which, as dependent on velocity, unfortunately cannot be tested, by virtue of the difficulty of giving a sufficiently high velocity to such comparatively heavy bodies as α-particles.

Rutherford's model of the hydrogen atom contains only a single electron and a single *positive electron*. But a hydrogen atom has the mass of 1,835 electrons, according to the best estimates, and therefore the positive electron is equivalent on this view, in spite of the identity of charge, to 1,834 electrons in mass. Since the radius of the electron is of order 10^{-13} cm., that of the positive electron is of order $\frac{1}{1}10^{-16}$ cm., and this greater concentration is responsible for the increase of mass.

Most of the properties of the hydrogen atom, and, in fact, all that are explicable on this basis, would, however, remain unchanged if the nucleus of the atom were not a positive electron, but an aggregate of nuclei and electrons whose total charge amounts to $+e$. There are many indications that this is a more correct view, and some of them may be reviewed briefly. In the first place, there is the modern chemical

theory of isotopic elements, introduced by Fajans, Soddy,* and Fleck, apparently supported by the system which it introduces into the arrangement of the products of radioactive elements in the periodic table, and by the probable existence of two varieties of neon with nearly the same atomic weight, discovered originally by Sir J. J. Thomson, and further investigated by Aston.† The position of an element in the periodic table, in this theory, depends not on its atomic weight, but on its *atomic number*, which, if not identical with, is certainly closely related to the charge on the atomic nucleus. Elements whose nuclei contain quite different aggregates of positive and negative electrons, and, therefore, different masses, are for all practical purposes identical if the aggregate in the two cases have the same resultant positive charge That these atomic numbers have a real significance is proved conclusively by the work of Moseley.‡ If this theory be accepted, we may inquire at what point in the periodic table this nuclear structure may be supposed to start. The existence of neon and metaneon, if they are actually isotopes, as seems necessary, shows that it has commenced long before the elements recognised as radioactive are reached. A favourite supposition is that it commences after helium, for all known α-particles appear to be nuclei of the helium atom. Such particles might, therefore, be the simple units which go to the making of a compound nucleus. But the existence of hydrogen, a simpler element, precludes the possibility that the helium nucleus is the simplest which can enter into such structures, and although suggestions have not been lacking that the nuclei of all elements are composed of hydrogen and helium nuclei, no experimental evidence of the existence of these hydrogen nuclei has been derived hitherto from the phenomena of radio-activity. In any case, helium nuclei cannot be the simplest forms of positive electricity, and we are tempted to suppose that complexity of structure extends throughout the nuclei of the terrestrial elements, and probably even to hydrogen. For the experiments of MM. Buisson and Fabry§ have shown that the secondary spectrum of hydrogen is largely due to atoms, and even by Bohr's method || we cannot explain the existence of

* "The Radio-Elements and the Periodic Law," Longmans, Green & Co., 1914.

† British Assoc. Report, Birmingham, 1913.

‡ "Phil. Mag.," 1913, VI., 26, p. 1024, Royal Society Discussion, 1914.

§ "Journal de Physique," June, 1912.

|| "Phil. Mag.," July and Nov., 1913.

so rich a spectrum without a more complicated model atom, even if the complication does not extend beyond the confines of the nucleus.

The charge on the nucleus of an atom is roughly, but in the case of the lighter elements only very roughly, proportional to the atomic weight. If these elementary nuclei were simple charges and not structures, some law of a simple character should connect these two quantities. It is precisely in the simplest elements that no such law can be found, even approximately. It seems necessary to believe that their nuclei are aggregates of charges, for which each individual member follows a simple law of mass which is hidden by the varying numbers of them in the aggregates. In heavier elements with very complex aggregates, the mass becomes proportional very closely to the number of charges in the aggregate, and follows another simple law which hides the first.

It is now almost certain that this conception represents the facts. For Nicholson* has shown that the spectra of unknown origin in the nebulæ and solar corona can be interpreted with great accuracy by the vibrations of atoms of the Saturnian type containing 3, 4, 5, and 6 electrons rotating round simple or non-structural nuclei of charges $3e$, $4e$, $5e$ and $6e$. The masses of these nuclei are at the same time determined from the spectra, in terms of the mass of the electron, and the ensuing atomic weights of the substances are, as expected, not those of known elements. For example, the nucleus $6e$ leads to an atomic weight 2·9, and MM. Bourget, Buisson and Fabry,† by applying their interference method to some of its lines in the nebular spectrum, have verified this value, showing at the same time, by accurate interference measurements of the lines, that they are found in the spectrum of no known element, and must be accredited to a new one The dissimilarity between the representations of these spectra by formulæ and the corresponding representations for the spectra of terrestrial elements points to an extreme degree of simplicity in these elements, although some are heavier than hydrogen But the most significant fact is that their calculated nuclear masses follow a simple law very accurately. They are proportional to n^2, where ne is the charge on the nucleus, and this is a natural law to expect—if they are really single portions of positive

* Monthly notices of Royal Astron. Soc., 1913-1915.
† "Journal de Physique," May, 1914.

electricity, concentrated into spheres of the same size—as the ordinary formula for mass indicates.

Now, according to this theory, the mass of a *positive electron* of charge e is only $2 \cdot 9/6^2$, or 0·081 of that of a hydrogen nucleus. Its radius is therefore $1/(1{,}835 \times 0 \cdot 081)$, or about 1/150 of that of an electron—or 13 times Rutherford's estimate. To obtain the mass of any *simple* positive charge ne, we use the expression $0 \cdot 081 n^2$, which rapidly becomes large as n increases. If $n=7$, we already have an atom as heavy as that of helium, and if $n=14$, as heavy as that of oxygen. But the atom of hydrogen must have a complex nucleus, and even its atomic weight is not that of a single particle of positive electricity.

Recent work on radio-activity, and on the effects of atoms in scattering α-and β-particles has also led to the conclusion that α-and β-particles come from the nucleus of the atom, which is accordingly complex. The older theory that β-particles came from an inner ring of electrons in orbital motion has been shown to be untenable on the theoretical side,* for such rings cannot exist in this type of atom, whether the atom is subject to ordinary dynamics, or the type of analysis used by Bohr † in his theory of the hydrogen spectrum. We must conclude, therefore, that, on all counts, the nuclei of the elementary atoms are structures of positive and negative electrons packed tightly together.

The tightness of this packing is instanced by the fact that one deduction made by Rutherford from his experiments, and from the analysis of Darwin, is that the law of action from the nucleus is still that of the inverse square at distances comparable with 10^{-12} cm., or with 10 times the radius of an electron. An assemblage of particles, each individually exerting this law of force, would give a different law on the whole unless their distances apart were small compared even with 10^{-12} cm. The distances apart are, therefore, of the order of the radius of an electron, and in the case of positive electrons, probably still smaller. In these circumstances, an important question arises, for we know from general considerations alone that the joint mass of two charges close together cannot be quite the sum of the individual masses, if all mass is really electromagnetic. The interaction of their fields must be taken into account; some mode of calculation of the mass of a compound nucleus is required. In the next section this problem is treated

* Nicholson, "Phil. Mag.," April and July, 1914.
† *Loc. cit.*

mathematically. The mass is derived in the usual way by setting the charges into motion with a small velocity, and working out the magnetic energy of their field. For simplicity, they are made to move in their common line with equal velocities.

Mass-formula for Two Charges.

If a small charge e moves along the axis of z in a co-ordinate system, the components of magnetic force (α, β, γ) which it produces at a point in external space are

$$\alpha=\frac{eu}{c}\frac{\partial}{\partial y}\left(\frac{1}{r}\right), \quad \beta=-\frac{eu}{c}\frac{\partial}{\partial x}\left(\frac{1}{r}\right), \quad \gamma=0, \quad . \quad . \quad . \quad (1)$$

where u is its velocity. The charge is at the origin at the instant considered, and (α, β, γ) is the magnetic force at a point (x, y, z) at distance r.

If the charge is at the point (o, o, c) on the z axis, the magnetic forces at an external point (x, y, z) are

$$\alpha=-\frac{euy}{cr^3}, \quad \beta=\frac{eux}{cr^3}, \quad \gamma=0,$$

where $r^2=x^2+y^2+(z-c)^2$.

If there are two such charges on the axis of z, each moving along it with velocity u,

$$\alpha=-\frac{uy}{c}\left(\frac{e_1}{r_1^3}+\frac{e_2}{r_2^3}\right), \quad \beta=\frac{ux}{c}\left(\frac{e_1}{r_1^3}+\frac{e_2}{r_2^3}\right), \quad \gamma=0, \quad . \quad . \quad (2)$$

where r_1 and r_2 are the distances from (x, y, z) to the two charges. The resultant magnetic force is H, where

$$\mathrm{H}^2=\frac{u^2}{c^2}\left(\frac{e_1}{r_1^3}+\frac{e_2}{r_2^3}\right)^2(x^2+y^2), \quad . \quad . \quad . \quad . \quad (3)$$

and the magnetic energy is

$$\iiint\frac{\mathrm{H}^2}{8\pi}dxdydz=\frac{u^2}{8\pi c^2}\iiint\left(\frac{e_1}{r_1^3}+\frac{e_2}{r_2^3}\right)^2(x^2+y^2)dxdydz, \quad (4)$$

the integrations extending throughout the space external to both charges. This is to be identified in the usual manner with $\frac{1}{2}mu^2$, where m is the *mass* of the pair of charges. The mass of the first charge moving alone is

$$\frac{e_1^2}{4\pi c^2}\iiint\frac{x^2+y^2}{r_1^6}dxdydz=\frac{2}{3}\frac{e_1^2}{a_1c^2}, \quad . \quad . \quad . \quad . \quad (5)$$

if a_1 is its radius, by a well-known calculation. Similarly

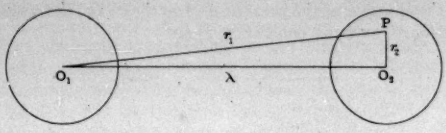
$$\frac{e_2^2}{4\pi c^2}\iiint\frac{x^2+y^2}{r_2^6}dxdydz=\frac{2}{3}\frac{e_2^2}{a_2c^2}. \quad . \quad . \quad . \quad . \quad (6)$$

if a_2 is the radius of the second charge. But the mass of the pair is

$$\frac{2}{3c^2}\left(\frac{e_1^2}{a_1}+\frac{e_2^2}{a_2}\right)+\frac{e_1e_2}{2\pi c^2}\iiint\frac{x^2+y^2}{r_1^3r_2^3}dxdydz. \quad . \quad . \quad (7)$$

The last integral is not zero, and it represents the mutual effect of their fields. Its value must now be calculated, and it will be referred to as the " mutual mass " of the two charges.

Consider now the order of magnitude of the integral when taken throughout one of the charges, the distance between their centres being λ, as in the figure. Then r_1 lies between the



values $\lambda\pm a_2$, and the integral between the values

$$\frac{e_1e_2}{2\pi c^2}\frac{1}{(\lambda\pm a_2)^3}\iiint\frac{(x^2+y^2)}{r_2^3}dxdydz,$$

which becomes when transformed to spherical polar co-ordinates of origin O_2,

$$\frac{e_1e_2}{2\pi c^2}\frac{1}{(\lambda\pm a_2)^3}\int_{r=0}^{a_2}\int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi}r_2\sin^2\theta d\phi d\theta dr_2,$$

or, on reduction,

$$\frac{2}{3}\frac{e_1e_2}{c^2}\frac{a_2^2}{(\lambda\pm a_2)^3}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

For two equal charges actually in contact, this contribution would be as important as the mass of each individual charge. But its relative importance is determined by the ratio $(a_2/\lambda)^2$, and if λ is even so small as $5a_2$, a formula which neglects it would be correct within about 1 per cent.

In a complex nucleus, we may suppose, for reasons to be given later, that the distances between the various α-particles

must be nearly of this order, and we may then regard this integral as negligible. In other words, instead of taking the integral in (7) over all space external to both charges, we may take it over *all* space, either external or internal. The work of calculation is thereby greatly simplified. But if the charges are closer together this procedure cannot be justified.

Let, then, λ be the distance between the charges, and let $r_1=r$. If we take 0_1 in the figure as the origin of polar co-ordinates (r, θ, φ) of any point in space,

$$r_2^2=r^2+\lambda^2-2r\lambda\cos\theta,$$

$$dxdydz=r^2\sin\theta d\varphi d\theta dr,\ x^2+y^2=r^2\sin^2\theta,$$

and the mutual mass is

$$m_{12}=\frac{e_1e_2}{2\pi c^2}\int_{r=0}^{\infty}\int_{\theta=0}^{\pi}\int_{\varphi=0}^{2\pi}\frac{r\sin^3\theta d\varphi d\theta dr}{(r^2+\lambda^2-2r\lambda\cos\theta)^{\frac{1}{2}}}$$

$$=\frac{e_1e_2}{c^2}\int_0^{\infty}\int_0^{\pi}\frac{r\sin^3\theta dr d\theta.}{(r^2+\lambda^2-2r\lambda\cos\theta)^{\frac{1}{2}}}$$

If $\cos\theta=\mu$, $r=\lambda\nu$,

$$m_{12}=\frac{e_1e_2}{\lambda c^2}\int_0^{\infty}\nu d\nu\int_{-1}^{1}\frac{(1-\mu^2)d\mu}{(1-2\mu\nu+\nu^2)^{\frac{1}{2}}}\quad\ldots\ldots\quad(9)$$

The double integral is entirely numerical, so that the mutual mass is of the form ae_1e_2/λ, where a is a pure number, and λ is the distance between the centres of the charges.

Now,

$$\int_0^{\infty}\frac{\nu d\nu}{(1-2\mu\nu+\nu^2)^{\frac{1}{2}}}=\int_0^{\mu}\frac{\nu d\nu}{[(\mu-\nu)^2+(1-\mu^2)]^{\frac{1}{2}}}+\int_{\mu}^{\infty}\frac{\nu d\nu}{[(\nu-\mu)^2+(1-\mu^2)]^{\frac{1}{2}}}$$

In the first of these integrals, write

$$\mu-\nu=(1-\mu^2)^{\frac{1}{2}}\tan\varphi,$$

where φ ranges from 0 to $\sin^{-1}\mu$, and in the second,

$$\nu-\mu=(1-\mu^2)^{\frac{1}{2}}\tan\psi,$$

where ψ ranges from 0 to $\pi/2$, and we obtain

$$\int_0^{\infty}\frac{\nu d\nu}{(1-2\mu\nu+\nu^2)^{\frac{1}{2}}}=\frac{1}{1-\mu^2}\int_0^{\sin^{-1}\mu}(\mu\cos\varphi-\sqrt{1-\mu^2}\sin\varphi)d\varphi$$

$$+\frac{1}{1-\mu^2}\int_0^{\pi/2}(\mu\cos\psi+\sqrt{1-\mu^2}\sin\psi)d\psi.$$

These integrals are easily evaluated, becoming respectively $\mu+\sqrt{1-\mu^2}$ and $1-\sqrt{1-\mu^2}$, and accordingly

$$\int_0^\infty \frac{\nu d\nu}{(1-2\mu\nu+\nu^2)^{\frac{1}{2}}} = \frac{1}{1-\mu^2}\{\mu+\sqrt{1-\mu^2}+1-\sqrt{1-\mu^2}\} = \frac{1}{1-\mu}. \quad (10)$$

Thus, from (9)

$$m_{12} = \frac{e_1e_2}{\lambda c^2}\int_{-1}^{1}(1-\mu^2)\frac{d\mu}{1-\mu} = \frac{e_1e_2}{\lambda c^2}\int_{-1}^{1}(\mu+1)d\mu = \frac{2e_1e_2}{\lambda c^2}. \quad (11)$$

The mass of two charges e_1, e_2 of radii a_1, a_2, at distance λ apart is therefore

$$\frac{2}{3c^2}\left(\frac{e_1^2}{a_1}+\frac{e_2^2}{a_2}+\frac{3e_1e_2}{\lambda}\right), \quad . \quad . \quad . \quad . \quad . \quad (12)$$

and the proximity of electrical charges destroys the conservation of mass. The mass is increased if two charges of like sign are brought closer, but decreases if they are of opposite sign. When they are *very* close together, this formula ceases to be applicable, but it can be used so long as λ is not greater than about five of their radii.

We can at once extend the result to any number of charges in proximity to one another. If e_r and e are two of them at distance λ_{rs} apart, the total mass is

$$\frac{2}{3c^2}\left\{\Sigma\frac{e_r^2}{a_r}+3\Sigma\Sigma\frac{e_re_s}{\lambda_{rs}}\right\}.$$

Consider, for example, an atom whose nucleus consists of a simple positive charge of amount ne, and a positive charge, also simple, of amount $2e$—the form assumed by Rutherford for the α-particles—together also with two electrons. Such an elementary atom would have perfect isotopy with a simpler element whose nucleus consisted *only* of a positive charge, ne. It could change into this simpler element by the emission of one α-particle and two β-particles. Such cases, in which an element returns, after two or perhaps three operations, to the same place in the periodic table, are of frequent occurrence in radioactivity, according to the theory of isotopes. The atomic mass of the element before these emissions is made up of (1), the mass m of the external electrons, n in number, rendering the atom neutral, (2) the mass m_1 of the nucleus ne, (3) the masses m_3 and $2m_4$ of the α-particle and the two nuclear electrons taken individually, and (4) the *mutual* masses of (3) and (2). After the emissions there are left only (2) and (1).

The radio-active change has, therefore, decreased the mass of the atom by m_3+2m_4, together with the mutual masses. We may neglect the individual masses of a small number of electrons like 2, as is ordinarily done in calculations. The mutual masses of the combination become, if λ is the mean distance between the centres of the components,

$$\frac{2e^2}{\lambda c^2}\{2n-(n+n)-(2+2)+1\}=-\frac{6e^2}{\lambda c^2},$$

independent of n. The first term is the mutual mass of nucleus and α-particle, the second of nucleus and each electron, the third of α-particle and electrons, and the fourth of the two electrons.

The loss of mass by the emissions is therefore $m_3-\frac{6e^2}{\lambda c^2}$, and not m_3, the mass of a helium atom, as ordinarily assumed. The magnitude of the correction is $9ma/\lambda$, where m and a are the mass and radius of an electron, and this is only comparable with the mass of a single electron.

Atomic weight calculations for radioactive products are not therefore seriously affected if the element ultimately ends as an isotope of its first form. But if it changes its position in the table the case is different. For example, the total passage from radium to lead involves a change of 10 in the atomic number, so that, on the whole, an amount $20e$ of positive electricity has left the nucleus in the form of 5 α-particles. In the radium atom the total mass is that of the lead nucleus and its attendant external electrons, of the five individual α-particles, and the mutual masses of the members of this group. Neglecting a few electrons, and also mutual masses involving the distant external electrons, we find that the absolute mass of a radium atom should exceed that of a lead atom by

$$5\mathrm{A}+\frac{5(5-1)}{2}\frac{2}{\lambda c^2}(2e)^2+10ne^2\cdot\frac{2}{\lambda c^2},$$

where A is the mass of a helium atom, n is the atomic number of lead—about 103—and λ is the average distance apart of the components in a radium nucleus. The factor $5(5-1)/2$ expresses the number of ways of grouping the α-particles in pairs and the last term is the mutual mass of the lead nucleus and the α-particles. The expression becomes

$$5\mathrm{A}+2040\frac{e^2}{\lambda c^2}.$$

If a is the radius of an electron, the mass of a hydrogen atom is $\frac{2}{3}\frac{e^2}{ac^2}\times 1835$. The decrease of atomic weight in passing from radium to lead would, therefore, be, since 3·994 is the atomic weight of helium,

$$5(3\cdot994)+\frac{2040\times 3}{2\times 1835}\frac{a}{\lambda}=19\cdot97+1\cdot65\frac{a}{\lambda}.$$

The atomic weight of lead derived from radium should, therefore, be, if we use Hönigschmid's determination of the atomic weight of radium (225·95),

$$205\cdot98-1\cdot65\frac{a}{\lambda},$$

where λ is a rough mean of the distances between α-particles in the atom, and a is the radius of the electron.

We can make a similar calculation for lead derived from thorium, taking $n=116$, and allowing for 6 α-particles instead of 5. The result is, if λ is the same as before, and if the atomic weight of thorium is 232·4,

$$208\cdot4-2\cdot40\frac{a}{\lambda}.$$

The accepted value for the atomic weight of ordinary lead is 207·10, and it is regarded as a mixture of leads from both sources. Actinium as a source is relatively unimportant. If we suppose the mixture to be in equal proportions, then on taking the mean,

$$207\cdot10=207\cdot19-2\cdot02\frac{a}{\lambda},$$

or,

$$\frac{\lambda}{a}=2\cdot5,$$

so that the average distance between the components of a nucleus is about $2\frac{1}{2}$ times the radius of an electron. As we are uncertain about the proportions, however, this calculation is unconvincing.

But Soddy has just given* a careful determination of the atomic weight of thorite lead, of which a large quantity has been isolated. The value is 207·64, and we may suppose that

* Vide, *e.g.*, "Nature," Feb. 4. 1915.

lead from radium is here absent. Applying the formula, therefore,

$$208{\cdot}40 - 2{\cdot}40\frac{a}{\lambda} = 207{\cdot}64,$$

or,

$$\frac{\lambda}{a} = 3,$$

almost exactly. This value therefore compels the average distance between the α-particles to be of the same order as the radius of an electron. Under this condition, the formula we have developed ceases to give the mutual mass of an electron and a positive charge. But it would still remain valid for positive charges, on account of their smaller size, and since in the passage from thorium to lead, positive charges are mainly in question, some reliance can be placed on the result. We have reached the conclusion, therefore, that if the atomic weight of thorite lead has the value found by Soddy, and if the higher value is not due to an unsuspected impurity, *the α-particles in a thorium atom have a mean distance apart comparable with the radius of an electron.* The importance of this conclusion in connection with atomic structure is evident, and on the supposition that all mass is electromagnetic, it is impossible to evade it.

In particular, it is a valuable indication that the nucleus of the atom is actually of very small dimensions and perhaps not very much greater than the radius of an electron. But it must be decidedly larger than Rutherford's estimate.

At the same time, these results appear to preclude any connection between the chains of development of uranium and thorium. For the atomic weight of lead from radium should be about 205·43, different from that of thorite lead by 2·21. If the chains are connected, they can only differ in their final products, the two leads, by about a multiple of 4 in atomic weight, unless we admit the existence of a new type of α-particle. The effect of mutual mass is not significant enough, *in the case of isotopes*, as we have seen, to alter this number 4. Thus, if lead is the ultimate product of radium, *all* the α-particles have been observed experimentally. On the supposition that ordinary lead has its origin in thorium and in radium, we can work out, from its atomic weight, the proportions of the two components. Thus, if there are x parts of thorium lead to one of radium lead,

$$207{\cdot}64x + 205{\cdot}43 = (x+1)(207{\cdot}10),$$

or $x=3$.

Thus, $\frac{3}{4}$ of the lead should arise from thorium. This is a point which could be tested on the basis of the present terrestrial distribution of thorium and uranium.

Many other questions are suggested by the calculation of the mutual mass of two charges. If, for example, charges really existed in the atom, closer to the nucleus than the outer ring of radius 10^{-8} cm., the mutual mass would make an important difference in their equations of motion, and would show its effect on the numerical constants of their spectra. Such an effect must be brought into any theory which involves charges moving in a small radius. But at a radius 10^{-8}, the effect is not appreciable with the order of accuracy which can be attained.

Apart from the deduction of the formula, which is the main object of the Paper, and the emphasis which must be laid on this question of mutual mass, almost completely overlooked hitherto in all theoretical discussion of the data of radioactivity, this Paper will have served its purpose if it succeeds in drawing attention to the great contributions to our knowledge of atomic structure, and more particularly nuclear structure, which can be given by some simple experiments, such as the precise determination of the atomic weights of lead from different sources, and by as many methods as possible. The actual conclusions in the last section are admittedly dependent on controversial points, for they involve the not generally accepted doctrine that the atomic weights of thorite lead and ordinary lead are different. Again, they involve the conclusion of Rutherford and Darwin that the law of force even at a distance of 10^{-12} cm. from the centre of a nucleus is still that of the inverse square. This conclusion, like the calculations in the Paper, would be overthrown if it could be proved that appreciable magnetic forces had their seat in the nucleus of an atom, as one school of physicists maintains. But the present calculations are the logical outcome of the suppositions just mentioned. Sir Ernest Rutherford, in the discussion on the structure of the atom at the Australian meeting of the British Association, proposed to leave nuclear structure to the next generation. It seems, on the other hand, to be vital to any further progress in our knowledge of atoms. The results already attained in the Paper are strongly in accord with the nucleus theory, since "mutual mass" is, on this theory, apparently capable of explaining the discrepancies which are observed in atomic weights.

ABSTRACT.

The Paper contains a mathematical deduction of a simple formula for the combined mass of two electrical charges when in proximity to each other. This mass is not the sum of their individual masses when far apart, if it be supposed that all mass of positive electricity, like that of electrons, is of electromagnetic origin. Applications are made of the formula to questions of atomic constitution and of radio-activity. A discussion is given of the evidence leading to the conclusion that the nuclei or cores of positive electricity in atoms are complex structures of electrons and even smaller positive nuclei. On this basis, emission of an α-particle by an atom does not decrease its atomic mass by 4, a correction being necessary for the "mutual mass" of the α-particle and the rest of the core. Estimates of the magnitude of this correction, in the case of radium and thorium passing into lead by the emission of particles, are given. From the value given by Soddy for the atomic weight of thorite lead we can deduce the average distance apart of the components in a radium nucleus. It is of the same order as the radius of an electron. Suggestions of further interesting applications of the precise formula for mutual mass are also contained in the Paper.

DISCUSSION.

Dr. H. S. Allen said that, in the model suggested by Sir E. Rutherford, the atom consists of a concentrated positive charge of extremely minute size surrounded by electrons revolving at different distances from the nucleus. In earlier Papers Prof. Nicholson has shown that this is mathematically impossible, and that all the external electrons must either form a single ring or rings in parallel planes. The Rutherford atom is then reduced to such extreme simplicity that it becomes incapable of explaining the complex facts of physics and chemistry. It cannot, for example, give an explanation of the complicated series of lines actually observed in the spectra of elements other than hydrogen, or of the secondary spectrum of hydrogen itself. Again, such an atom should have a magnetic moment simply proportional to the number of electrons in the ring, for the nucleus is too small to produce by rotation, or otherwise, any appreciable magnetic moment. The results of the present Paper appear to prove that the nucleus of an ordinary atom cannot have the small dimensions previously assigned to it. It seems certain that the nucleus cannot be so minute if we assume that there exist in it α and β particles as such. The only reason for assigning an extremely small diameter to the nucleus of a heavy atom is to account for the wide-angle scattering of α particles, but in the discussion of this question no attention was paid to the possible action of magnetic forces on the moving particle. If these exist the trajectories become complicated, but it is probable that the scattering could then be accounted for without necessitating such a near approach to the centre of the atom. We are thus led to the view that the central portion of the atom may contain α and β particles in orbital motion which would set up a magnetic field. But as the velocities must presumably be less than that of light, the radius of such a *magnetic core* must be considerably greater than that of the simple nucleus of Rutherford.

XIX. *The Estimation of High Temperatures by the Method of Colour Identity.* *By* CLIFFORD C. PATERSON *and* B. P. DUDDING, *A.R.C.Sc.* (*From the National Physical Laboratory.*)

Synopsis.

1. Preliminary experiments are described on the method of "colour identity" adapted to the estimation of the temperature of incandescent substances such as metal or carbon radiating in the open; by this method the "true" temperature of certain bodies as distinct from their "black body" temperatures can be arrived at with a very fair degree of accuracy.

2. By the colour identity method the total luminous radiation (white light) from a black body is made identical in colour with that from the incandescent metal under examination by adjusting the temperature of the black body until there is colour identity in the field of a Lummer Brodhun photometer.

3. Comparisons are made of the results so obtained with those obtained by other methods, and the colour identity method is shown to give the correct result for melting platinum.

4. Formulæ are deduced, based on the fundamental theories of energy radiation and the sensitivity of the eye, connecting the temperature of carbon and tungsten filaments with their lumens per watt, and it is shown that these expressions hold from the lowest to the highest values of lumens per watt.

5. It is shown that the colour identity method of determining filament temperatures is practically independent of the cooling at the ends of the filaments of ordinary lamps.

6. An explanation is given of the principal factors and limitations of the colour identity method in which it is shown that accurate results should be obtained so long as the bodies under consideration act as "grey" bodies throughout the visible spectrum, and that there will be a tendency to error to the extent that they depart from the grey body condition in the *visible spectrum*.

7. The colour of the radiation from melting platinum is shown to be the same as that from a carbon filament lamp operating at $2{\cdot}6_5$ lumens per watt, or $4{\cdot}7_5$ watts per mean spherical candle, or approximately 3·8 watts per mean horizontal candle.

The work described in this Paper is not in the nature of a complete investigation of the subject. It had for its original

object the determination of the colour of the light from molten platinum under the open radiation conditions which prevail in the realisation of the Violle standard of light A good primary standard of light must not only be constant and accurately reproducible, but the colour of its light should approximate to that of the sources which are ordinarily used in practice, so that large colour differences will not be involved in the photometric measurements for which such a standard is used. The object for which the investigation was started was completed over two years ago, but the progress of the work indicated some unexpected phenomena which it was intended to investigate further. Pressure of other work has up to now prevented this being done, and the authors desire at this stage to publish this preliminary note on the subject. The accuracy of the work is the accuracy of preliminary experiments in which all reasonable precautions have been taken. Values given for temperature certainly have not an absolute accuracy of more than 1 or 2 per cent., but the methods described are capable of a higher precision, and this will undoubtedly be attained in the fuller investigation which it is intended to undertake.

General Discussion.

Optical pyrometry is almost exclusively concerned with the *intensity* of the light emitted by a luminous body in any given wave-length. The colour of the light thus dealt with is fixed by the wave-length or wave-lengths chosen for the measurements, and colour differences do not occur.

In ordinary photometry the sum of the intensities of the light emitted by a source in all wave-lengths over the visible spectrum is compared against the sum of the intensities of the light emitted by another source. The radiation from each source has thus a composite colour whose characteristics will depend on the relative intensity of the light in each wave-length. Although both sources may radiate according to the law of a black body, if there should be a difference of temperature between them the composite colour or hue of the radiations from the two bodies will differ, and it becomes necessary to compare intensities which are not of the same colour. For most solid radiators the colour of the light is a perfectly definite quality, and forms a criterion of the state of incandescence of such bodies. Most bodies are more or less selective in their radiation, but there is a certain group, consisting mainly of metallic substances, which although appa-

rently selective in favour of the visible spectrum *as a whole*, emit light throughout that spectrum without any appreciable deviation from the distribution to be found in the visible spectrum of a black body.* For instance, consider a tungsten filament adjusted to a suitable temperature, and compared spectrophotometrically against a carbon filament. The one is mainly selective in favour of the visible spectrum as a whole, and the other acts in this respect as a black body. The spectrophotometer, dealing only with the visible spectrum, cannot detect any relative difference between the two at different wave-lengths throughout the portion of the spectrum with which it deals, and thus a comparison of the total visible radiation is possible with an ordinary photometer, exact identity of colour being obtainable. That is to say, these substances virtually radiate as "grey" bodies, as far at least as the visible spectrum is concerned, and it is this close approximation to grey body radiation in the visible spectrum which lies at the root of the method discussed in this Paper. Hence, identity of colour can be obtained not only when comparing one tungsten lamp against another, but also when comparing a tungsten lamp against a black body. If the temperature of these bodies is pushed to an extreme value a very slight difference of colour is perceptible at the point where the colour balance is closest, but such differences are too small to prevent an observer obtaining consistent results in judging the colour balance between two radiations.

A comparison of colour is made similarly to photometric comparisons of intensity. The current in the comparison lamp is varied so that the colour of the light fluctuates on both sides of the mean, first inclining to be redder and then to be bluer than the light from the test source. The current in the lamp is then readily determined at which the observer judges the colour balance to occur. It must be remembered that in these comparisons it is the hue of so-called white light which is under consideration, and not that of spectral or other colours.

The colour identity method depends on the combined effects of the light emitted in all wave-lengths in the visible region. If the intensity is relatively greater at the red end than at the blue end, the hue of the resulting radiation will tend to be red, and vice versa. The radiation from a black body at 1,750°C.

* Coblentz, "Radiation Constants of Metals," "Bull." B.S., Vol. V. p. 359. Hyde, "Selective Emission of Incandescent Lamps," "Trans.," Ill. Eng. Soc., 1909.

has a definite hue depending on the relative proportions of the energy in the red, green and blue regions, and any other radiator emitting light in the same *relative proportions* will have the same hue of radiation, no matter what the absolute intensity of the radiations. Thus, it is that the radiation from a grey body will be identical in hue with that of a black body, and compared on the colour identity basis the grey body will be given its true temperature. The optical pyrometer, on the other hand, only takes account of the relative *intensities* of the light from the black and grey bodies, and, therefore, estimates the temperature of the grey body at a value far below its true temperature. It follows, therefore, that the measure of the accuracy of the colour identity method is the extent to which bodies radiate as *grey* (*or black*) *bodies throughout the visible spectrum.*

Throughout this Paper the usual conception of a grey body is adopted—*i.e.*, one which, at any temperature, does not radiate as much energy in the various wave-lengths as a black body at the same temperature, but in any wave-length the intensity per unit area of the surface is a constant fraction of that of the black body in the same wave-length.

By a selective body is meant one in which the amounts of energy radiated in the various wave-lengths throughout the whole spectrum do not bear a constant proportion to those in the same wave-lengths for a black body at the same temperature.

Section 1 of this note deals with the establishment of electric sub-standards of colour, which are intended to serve for defining the colour of the radiation from any incandescent bodies compared against them, and so to fix the temperature of such bodies in terms of the temperature of a black body whose radiation is identical with theirs in colour.

Section 2 gives the determination of "colour identity" temperatures of carbon and tungsten glow lamps when burning at different efficiencies, and contains expressions for such efficiencies in terms of temperature based on Wien's equation for intensity of energy distribution and Nutting's equation for the sensitivity of the human eye.

Section 3 discusses the accuracy of such determinations, and deals with the "colour identity" temperature of platinum at the melting point, showing that even for a selective radiator such as platinum this temperature is a measure of the true

temperature of the platinum filament, although it is glowing under open radiation conditions. Filament temperatures for carbon and tungsten (vacuum and gas-filled) are also discussed.

Section 4 deals with the colour of the radiation from molten platinum in relation to the practical usefulness of the Violle standard of light.

1. *Electric Sub-standards of Colour for the Determination of Temperature.*

In spite of the fact that the device of colour comparison by means of a Lummer Brodhun photometer has been used for many years by various observers for obtaining equality of efficiency of glow lamps of the same type, it is not generally realised how easily and with what precision such colour com-

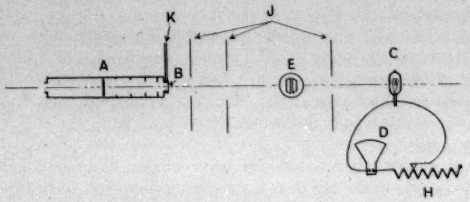


FIG. 1.—DIAGRAM SHOWING ARRANGEMENT OF APPARATUS USED TO OBTAIN THE RELATION BETWEEN THE CURRENT IN CARBON AND TUNGSTEN FILAMENT LAMPS, AND THE TEMPERATURE OF A BLACK BODY AT EQUALITY OF HUE OF THEIR RADIATIONS.

parisons can be made. Morris,* Stroud and Ellis employed this method in 1907, and extensive use has been made of it for investigating selectivity and other properties of radiating substances by E. P. Hyde,† with whom Cady and Middlekauff have sometimes collaborated. Hyde,‡ in discussing the question of colour identity and temperature (p. 40, *loc. cit.*), showed that a colour match with a black body might be regarded as indicating that the temperature of the black body was at

* Morris, Stroud and Ellis, "The Electrician," Vol. LIX., p. 584.

† Hyde, Cady and Middlekauff, "Selective Emission of Incandescent Lamps," Ill. Eng. Soc., New York, Vol. IV., 1909, p. 334. Hyde, "Physical Characteristics of Luminous Sources," Lectures, John Hopkins University, 1910. Hyde, "Radiation Laws for Metals," "Astrophys. Journ.," Vol. XXXVI., 1912, p. 89.

‡ Hyde, "The Physical Production of Light," "Journ." Franklin Inst., Vol. CLXX., 1910.

least as high or higher than that of the body compared against it, but he expressed the opinion (p. 39) that under the condition of colour identity two different radiators although with continuous spectra would not be at the same temperature.

A black body furnace electrically heated and capable of being raised to a temperature of 2,200°C., with a clear internal atmosphere, was kindly put at the disposal of the authors by Dr. J. A. Harker, F.R.S. The very excellent arrangements of this furnace need not be explained in detail here. The apparatus is shown diagrammatically in Fig. 1. A is a black body kept clear of fumes by a stream of nitrogen admitted at K.

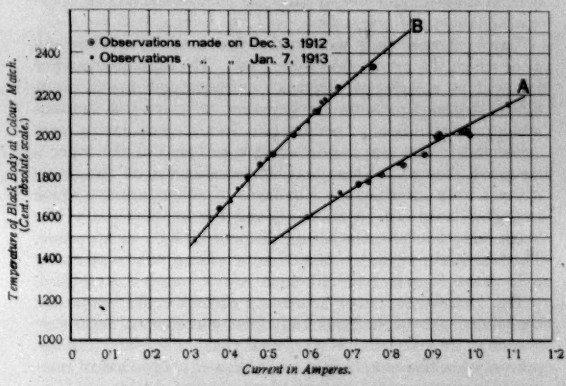


FIG. 2.—CARBON (A) AND TUNGSTEN (B) COLOUR STANDARDS.

Curves connecting the current in the lamps with the temperature of the black body whose radiation is identical in colour with that of the lamps.

B is a diaphragm with glass window, which permits only light from the centre of the incandescent surface to pass down the photometer bench, and J are screens to cut off extraneous light. C is a carefully seasoned electric lamp, the current through whose filament can be accurately measured by means of ammeter D. Between the two at E is a Lummer Brodhun photometer head. This photometer is not used to compare the *intensities* of the two sources of light A and C, but to determine when the hue of their radiations is identical. Two optical pyrometers were used of the Siemens and Féry types, by means

of which the temperature of the black body was determined before and after each colour determination by the photometer. Complete determinations were made on two separate occasions and the mean result taken—every photometer and pyrometer setting being made on each occasion by two observers. The furnace was first run at a relatively low temperature, and the electric heating adjusted so that it would remain at a constant temperature sufficiently long for both photometric and pyrometric readings to be taken. The hue of the light from the electric lamp C was varied by means of rheostat H until there was exact identity of colour in the photometer. In doing this it is, of course, necessary to place the photometer so that there is also equality of brightness.

Two or three settings were made by each observer before and after which the pyrometers were read. The same process was followed in a series of increasing temperatures up to about 2,200°C.—the maximum temperature to which the furnace was carried in these experiments. Two electric lamps were calibrated in this way, one having a carbon and the other a tungsten filament. These two lamps calibrated in the above manner form intermediate standards of colour against which any glow lamp can be matched, and the temperature of the black body fixed to which the colour of its light corresponds. The temperatures so determined are on the optical scale, using pyrometers for which the dominant wave length is $\lambda=0{\cdot}650\mu$.

Fig. 1 shows the curve connecting current in the lamps and the temperature of the black body for colour identity in the case of each of these lamps. It was found that the sensitivity of the process of colour matching is more than equal to that of temperature measurement by optical pyrometers.

Duplicates of these two lamps were then made for use in experiments where their continued employment at the higher efficiencies might affect their constancy.

2. "*Colour Identity*" *Temperatures Corresponding with Different Efficiencies.*

It is common for the specific consumption of glow lamps to be stated in terms of the watts per mean horizontal candle. The only rigorous way, however, is to state it in terms of watts per mean spherical candle or in lumens per watt. Throughout this Paper the latter designation is used, representing as it does the true measure of the ratio of the total light emitted to the

power supplied. In the table of results the approximate watts per mean horizontal candle of the glow lamps is also given, since this is the more familiar designation, but it is not rigorous on account of the varying ratios of mean horizontal to mean spherical candle-power to be found in different lamps. It is to be noted that in the determination of the total light emitted from the glow lamps the light which is obscured by the cap of the lamp is counted as being radiated and not absorbed, but in any case this light is less than 1 per cent. of the total for filaments of ordinary form.

The objects of the measurements are :—

(*a*) to find the relation between the various values of lumens per watt for tungsten and carbon filament lamps and the corresponding temperatures of a black body on the basis of colour identity ;

(*b*) to ascertain to what extent the temperature so measured represents those of the principal parts of the glowing filaments, having regard to the cooling effect of the filament supports ;

(*c*) to find laws connecting lumens per watt and corresponding "colour identity" temperature for tungsten and carbon filaments.

Evidence is given later in the Paper to show that there seems to be justification for the assumption that the colour identity method at any rate in certain cases gives within narrow limits a measure of the true filament temperature. This assumption is, therefore, made in what immediately follows here, and it will be seen that the results which follow from this assumption whilst not proving its validity, are in agreement with those of Forsythe, who determined temperatures by more orthodox methods.

(*a*) A number of carbon and tungsten filament lamps were selected for the measurements. The carbon lamps had both flashed and unflashed filaments. The tungsten lamps had squirted and drawn filaments of different diameters and lengths, so that the effect of the cooling of the ends by the leading-in wires if appreciable might be observed.

All the lamps were measured for lumens per watt at different voltages up to the highest they were capable of standing without deterioration. They were then compared for identity of colour against the colour standards, and in this way the temperatures of a black body were determined which corresponded with the various values of lumens per watt. The results are given in Table I., and plotted in Fig. 3.

TABLE I.

(Reduction factor=Mean spherical candle-power divided by mean horizontal candle-power.)

Carbon Filament Lamp (flashed) No. 1.
100 *volts*, 16 *candles*.
Reduction Factor=0·85$_5$.

Volts.	Watts per mean horizontal candle.	Lumens per watt.	Temperature of black body at identity of colour (Cent. absolute).
49·2$_5$	111·$_0$	0·09$_7$	1,515
59·5$_0$	38·$_6$	0·28$_4$	1,640
72·7$_0$	13·$_6$	0·77$_5$	1,755
82·7$_0$	7·9$_5$	1·35	1,865
92·5$_0$	5·0$_3$	2·14	1,960
103·0	3·7$_7$	2·85	2,055
110·0	2·5$_3$	4·14	2,120
130·0	1·5$_6$	6·85	2,250
135·0	1·3$_7$	7·7$_7$	2,300
140·0	1·2$_1$	8·8$_5$	2,325

Carbon Filament Lamp (flashed) No. 2.
100 *volts*, 16 *candles*.
Reduction Factor=0·89.

Volts.	Watts per mean horizontal candle.	Lumens per watt.	Temperature of black body at identity of colour (Cent. absolute).
65	20·$_2$	0·55$_5$	1,715
70	14·$_1$	0·80	1,775
75	10·$_2$	1·09	1,835
80	7·7$_0$	1·45	1,890
85	6·1$_3$	1·84	1,935
90	4·8$_2$	2·33	1,980
100	3·2$_1$	3·43	2,075
105	2·6$_9$	4·17	2,120
107	2·5$_2$	4·44	2,135

Carbon Filament Lamp (flashed) No. 3.
100 *volts*, 16 *candles*.
Reduction factor=0·85.

65	19·$_5$	0·55	1,720
70	13·$_8$	0·77$_5$	1,775
75	10·$_0$	1·06$_5$	1,835
80	7·6$_4$	1·40$_5$	1,890
85	6·0$_4$	1·77$_5$	1,935
90	4·8$_5$	2·20	1,985
100	3·3$_4$	3·21	2,070
105	2·7$_5$	3·83	2,115
107	2·6$_3$	4·07	2,130
115	2·0$_4$	5·25	2,200
120	1·8$_1$	5·91	2,240
130	1·4$_3$	7·50	2,310
135	1·2$_9$	8·58	2,345

Carbon Filament Lamp (flashed) No. 4.
200 *volts*, 16 *candles*.
Reduction factor=0·86$_5$.

140	18·$_2$	0·60	1,710
150	12·$_7$	0·85	1,770
160	9·0$_0$	1·18	1,835
180	5·2$_6$	2·09	1,955
200	3·2$_7$	3·41	2,070
210	2·6$_8$	4·05	2,120
220	2·2$_0$	4·95	2,165

Carbon Filament Lamp (unflashed) No. 5.
200 *volts*, 16 *candles*.
Reduction factor=0·82$_5$.

140	22·$_2$	0·47	1,710
150	15·$_2$	0·68	1,775
160	10·$_9$	0·95	1,835
180	6·1·$_4$	1·68	1,940
200	3·7$_8$	2·74	2,050
210	3·0$_8$	3·36	2,095
220	2·5$_4$	4·08	2,140

Tungsten Filament Lamp (Drawn) No. 6.
115 *volts*, 30 *watts*.
Reduction factor=0·79.

38·2	21·$_7$	0·46	1,640
47·4	11·$_1$	0·90	1,755
57·8	6·3$_5$	1·57	1,865
68·7	4·0$_3$	2·47	1,960
93·2	1·9$_2$	5·17	2,145
120·5	1·1	9·00	2,325

Tungsten Filament Lamp (Squirted) No. 7.
105 *volts*, 32 *watts*.
Reduction factor=0·79.

38·8	15·$_6$	0·63	1,640
48·2	9·3$_4$	1·06	1,755
57·2	5·4$_1$	1·84	1,865
67·0	3·5$_5$	2·78	1,960
90·5	1·7$_6$	5·66	2,145
117·0	1·0$_8$	9·54	2,325

Tungsten Filament Lamp (Squirted) No. 8.
105 *volts*, 30 *watts*.
Reduction factor=0·78$_5$.

37	14·7	0·72	1,710
40	11·$_6$	0·85	1,745
45	7·8$_5$	1·26	1,800
50	6·1$_6$	1·62	1,860
55	4·7$_8$	2·10	1,900
60	3·7$_8$	2·61	1,960
65	3·0$_6$	3·28	2,010
70	2·6$_0$	3·81	2,055
75	2·2$_2$	4·47	2,095
80	1·9$_1$	5·21	2,135
90	1·4$_3$	6·65	2,215
100	1·2$_6$	8·19	2,285
105	1·0$_9$	9·06	2,320

TABLE I.—*Continued.*

Volts.	Watts per mean horizontal candle.	Lumens per watt.	Temperature of black body at identity of colour (Cent. absolute).
Tungsten Filament Lamp (Squirted) No. 9. 105 volts, 60 watts.			
	Reduction factor = 0·78$_5$.		
40	12·8	0·77$_5$	1,710
45	9·8$_2$	1·01	1,775
50	7·3$_4$	1·35	1,835
55	5·7$_0$	1·74	1,880
60	4·5$_3$	2·18	1,920
65	3·7$_1$	2·67	1,970
70	3·1$_2$	3·17	2,015
75	2·6$_5$	3·73	2,055
80	2·2$_9$	4·37	2,100
90	1·7$_4$	5·70	2,175
100	1·3$_9$	7·12	2,245
105	1·2$_5$	8·06	2,280
110	1·1$_4$	8·66	2,315
115	1·0$_4$	9·51	2,340
120	0·94$_5$	10·4	2,375
125	0·88$_5$	11·2	2,395
130	0·82$_0$	12·1	2,435
135	0·76$_0$	13·0	2,460
Tungsten Filament Lamp (Drawn) No. 11. 200 volts, 20 watts.			
	Reduction factor = 0·78$_5$.		
80	13·5	0·73	1,720
90	9·7$_6$	1·02	1,770
100	7·2$_5$	1·36	1,835
110	5·5$_0$	1·80	1,885
120	4·4$_2$	2·23	1,935
130	3·5$_6$	2·75	1,970
140	3·0$_1$	3·28	2,020
150	2·5$_3$	3·87	2,060
160	2·1$_8$	4·4$_8$	2,105
180	1·7$_1$	5·8$_7$	2,170
200	1·3$_8$	7·2$_7$	2,245
210	1·2$_5$	7·9$_0$	2,270
220	1·1$_4$	8·6$_0$	2,320

Volts.	Watts per mean horizontal candle.	Lumens per watt.	Temperature of black body at identity of colour (Cent. absolute).
Tungsten Filament Lamp (Drawn) No. 10. 100 volts, 15 watts.			
	Reduction factor = 0·78$_5$.		
45	8·6$_9$	1·17	1,800
50	6·1$_5$	1·59	1,860
55	4·8$_0$	2·05	1,915
60	3·8$_8$	2·55	1,960
65	3·2$_0$	3·09	2,015
70	2·6$_5$	3·71	2,055
75	2·2$_6$	4·36	2,105
80	1·9$_5$	5·10	2,140
90	1·5$_0$	6·75	2,225
100	1·2$_1$	8·13	2,290
105	1·1$_1$	9·02	2,325
110	1·0$_0$	9·88	2,360
115	0·91$_5$	10·8$_2$	2,400
120	0·83$_5$	11·8$_3$	2,430
125	0·77$_5$	12·7$_6$	2,453
130	0·72$_3$	13·7$_3$	2,485
135	0·66$_5$	14·8$_5$	2,505
Tungsten Filament Lamp (Drawn) No. 12. 230 volts, 60 watts.			
	Reduction factor = 0·78$_5$.		
100	9·5$_9$	1·04	1,770
110	7·3$_5$	1·34	1,825
120	5·7$_0$	1·74	1,870
130	4·5$_4$	2·17	1,915
140	3·8$_0$	2·60	1,960
150	3·1$_9$	3·10	2,000
160	2·7$_4$	3·60	2,040
170	2·3$_5$	4·1$_9$	2,080
180	2·0$_5$	4·8$_1$	2,110
190	1·8$_1$	5·4$_6$	2,145
210	1·4$_6$	6·7$_8$	2,210
230	1·2$_2$	8·0$_8$	2,270
240	1·1$_3$	8·7$_7$	2,305
260	0·96$_6$	10·2	2,360

In obtaining the higher temperature values for plotting on Fig. 3, it is very useful to make use of a "watt-temperature" curve. The carbon lamp, for instance, cannot with safety be run for long periods at temperatures in the region of 2,000°C. If the watts be plotted against temperatures obtained by the identity of colour method the resulting curve will be found to be a logarithmic, and no deviation whatever can be detected

from such logarithmic over the range between the highest and the lowest observed values. This is illustrated in Fig. 4, in which the logs of temperature and watts have been plotted for both carbon and tungsten lamps Nos. 3, 5, 6, 8 and 12. It is

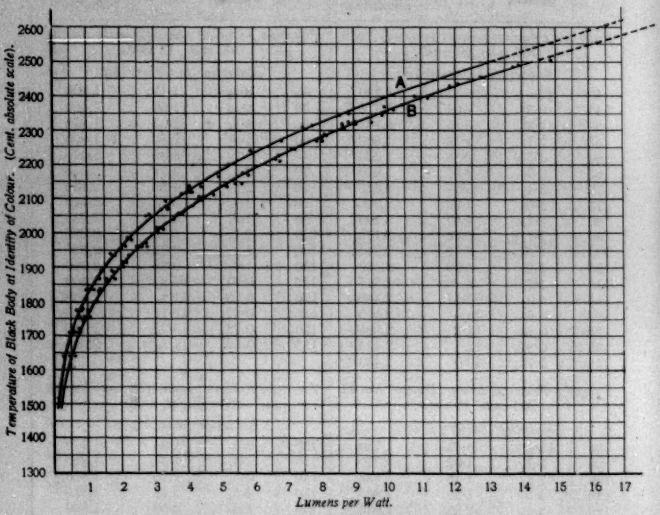


FIG. 3.—ORDINARY 100 AND 200-VOLT CARBON (CURVE A) AND TUNGSTEN (CURVE B) FILAMENT VACUUM LAMPS.

Curves connecting lumens per watt of the lamps with the temperature of a black body whose radiation is identical in colour with that of the lamps.

safe from a knowledge of the watts in any lamp to deduce intermediate temperature values from such a curve, so that the actual number of colour comparisons may be a minimum, and the burning period of the colour standard reduced.

The watt-temperature relation is given, for carbon filament lamps, by

$$W \propto T^{4\cdot58},$$

and for tungsten filament lamps by

$$W \propto T^{5\cdot1},$$

Considering again Fig. 3, the first point to notice is that no difference can be detected between the various carbon lamps tested or between the different tungsten lamps. Whether the carbon filaments are flashed or unflashed, and the tungsten filaments squirted or drawn, appears to leave unaffected the relation between lumens per watt and the corresponding "colour identity" temperature of a black body for either of these types.

It will, therefore, be seen that all the results may be taken as

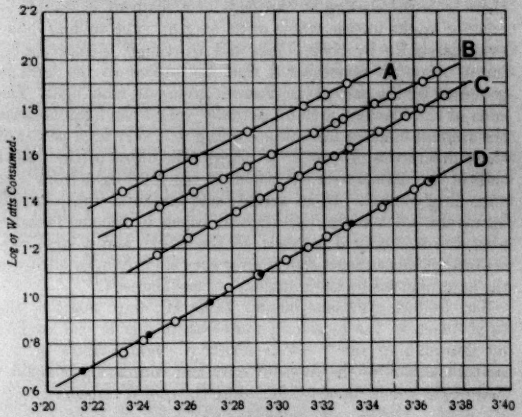


FIG. 4.

Curves connecting watts consumed by carbon and tungsten filament lamps, and the temperature of the black body whose radiation matches in colour that of the lamps.

A is for	Lamp	No. 5.	(Carbon filament.)
B	,,	,, No. 3.	(,, ,,)
C	,,	,, No. 12.	(Tungsten ,,)
D	,,	,, Nos. 6 and 8.	(,, ,,)

lying on two curves, one representing the carbon group and the other the tungsten group, and so closely do the points keep to the curves that very few observations lie more than 1 per cent. in temperature from the mean curve.

This implies that in all ordinary lamps of the same character (vacuum tungsten or carbon) the colour of the radiation from the whole filament, including the cooled ends, is the same for the same value of lumens per watt. Therefore, to considerable

accuracy it may be said, that a knowledge of the lumens per watt of a lamp implies a knowledge of the temperature of a black body whose radiation is the same in hue as that of the lamp.

(*b*) The following considerations show to what extent a temperature, determined as above, may be regarded as the temperatures of the main glowing part of the filament. If there were no cooling at the ends of a filament it would be equally bright for the whole of its length. The cooling, however, as shown by Hyde, Cady and Worthing* is appreciable, but it must be remembered that the colour of the light is governed by the part of the filament giving off most light. The ends of the filament give off actually very little light because the amount emitted falls off according to a very high power of the temperature (T^{12} to T^{20}). Hence the effect of the dulled ends of the filament on the colour of the total light from it, is exceedingly small. In an actual case† the total light emitted below the point where the filament began to become measurably dull was only 5 per cent. of the whole, and a large percentage of this amount differs only very slightly in colour from the light emitted by the remainder of the filament.

The following measurements, Table II., were made of the total effect on the measured temperature due to end cooling by determining the "colour identity" temperature of the central portion of the filament only and comparing it against that of the whole filament, including the cooled ends.

TABLE II.

Lamp.	Volts.	Temperature of black body for colour identity.	
		Whole filament.	Centre of filament.
No. 11	200	2,245	2,255
	110	1,885	1,898
	54	1,556	1,570
No. 9	105	2,280	2,290
	60	1,920	1,930
	40	1,710	1,722
Motor headlight. 16 volts, 50 watts	...	1,486	1,508
	...	2,014	2,043
	...	2,240	2,285

* Amer. Ill. Eng. Soc. "Trans." 6, pp. 238-257.
† *See* Hyde, Cady and Worthing. *Loc. cit.*

Lamp No. 11 was an ordinary 200 volt, 20 watt tungsten lamp, and No. 9 was rated for 100 volts, 55 watts; the difference of filament diameter and distance between supports were, therefore, as large as is usually met with in practice. The motor headlight filament was for 16 volts and 50 watts, and, therefore, represented an extreme case.

It will be seen from this that unless a very thick, short filament be taken with abnormal end cooling, the measured colour identity temperature will be that of the central bright portion of the filament, within about 1 per cent. Whilst in the extreme case of the headlight lamp it is of the order of 2 per cent. It is obvious that the cooling effect for carbon filament lamps is considerably less than for tungsten, and is, in fact, quite inappreciable.

It is thus clear that the colour identity method gives results which depend very closely on the temperature of the central portion of the filament. If it may be assumed that the method also gives the *true* temperature of lamp filaments, the figures in Table I. and Fig. 3 indicate the appreciable difference of efficiency existing between the carbon and tungsten lamps for the same temperature, and, therefore, establish the selectivity of the tungsten filament in favour of the shorter wave-lengths, a subject upon which much has been written, and which has been thoroughly investigated by Dr. E. P. Hyde.* This difference in efficiency would, if anything, be very slightly increased by taking into account the end cooling of the filaments, the tendency of which is to act in favour of the carbon lamp. Also, if the carbon filament is "greyer" than the tungsten filament in the visible region the apparent difference of efficiency will be increased.

(*c*) Referring to the curves shown in Fig. 3 and bearing in mind what has been said in the foregoing remarks, it becomes of interest to know if a relation connecting lumens per watt and temperature can be deduced from our knowledge of the phenomena involved, and especially to ascertain how nearly the experimental observations conform to such a relation deduced from theoretical considerations.

We have to consider, therefore, how the rate of dissipation of energy by a lamp filament, *i.e.*, the watts,† increases with a

* *See* Hyde. *Loc. cit.*

† The rate at which energy is radiated is power and is spoken of hereafter as radiant power.

rise in temperature, and also how the eye estimates the rate at which this energy is radiated.

The eye is only sensitive to a small portion of this energy, *i.e.*, that emitted in wave-lengths lying approximately between $0{\cdot}3\mu$ and $0{\cdot}8\mu$. Further, the eye does not appreciate the intensity of the energy radiation in any wave-length over this limited range in direct proportion to the amount radiated, but weights it according to its own peculiar sensitivity to energy of that wave-length. This appreciation of power by the eye is expressed in lumens which may be defined as the measure of the appreciation of the eye for radiant power.

An expression must, therefore, be found connecting lumens and the temperature of the radiating body both in terms of the power distribution throughout the visible spectrum and of the sensitivity characteristics of the eye.

The theoretical investigation of the problem thus subdivides itself naturally into three distinct parts :—

(*a*) The rate of energy dissipation of the radiator at any temperature.

(*b*) The quantitative distribution of this radiant power throughout the spectrum at any temperature, with special reference to that range of the spectrum over which the energy stimulates the sense of vision.

(*c*) The relative capacity of equal amounts of radiant power in different wave-lengths for stimulating vision, this being necessarily referred to the average or normal human eye.

(*a*) *Relation Between Watts and Temperature.*—Attention has been already drawn to curves showing the relation between the rate of dissipation of energy by a lamp and its temperature as measured by the colour identity method. Many lamps of ordinary dimensions have been examined, and in all cases the results can be expressed by an equation of the form

$$\text{watts} \propto T^m \text{ or } (\log W = \log D + m \log T), \quad . \quad . \quad . \quad (1)$$

m being $4{\cdot}_5$ to $4{\cdot}_6$ for carbon lamps and $5{\cdot}0_5$ to $5{\cdot}_2$ for tungsten lamps.

In no case has any appreciable deviation been observed from this logarithmic relationship for temperatures ranging from 1,700 deg. to 2,300 deg. abs.

This relationship is at once recognised as being identical in form with that ascribed to Stefan and Boltzman connecting the temperature and radiant watts of the ideal black body, m in the latter case being $4{\cdot}_0$.

(*b*) *Distribution of Radiant Power throughout the Visible Spectrum.*—In the case of the ideal black body, the radiant power in any wave-length of the visible spectrum at any temperature below 3,000°C. can be expressed according to the well-known law of Wien

$$E = C_1 \lambda^{-n} e^{-\frac{C_2}{\lambda T}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

E being the radiant power of wave-length λ at temperature T, C_1 and C_2 being constants, $n=5$ (for a black body).

Seeing that in the experiments described in the following section (3), in which the colour of the radiation of melting platinum was found to be identical with that of a black body operating at the same temperature, it is reasonable to assume that the power distribution, at least *over the range of the visible spectrum*, can be expressed by a formula of the above form. Lummer and Pringsheim found this condition to be closely fulfilled by radiators having the characteristics of platinum.

When the visible spectrum only is under consideration the values of C_1, C_2 and n can vary considerably without affecting the shape of the curve by an amount corresponding to a difference of temperature of 10°C. in the region of 2,000°C.

(*c*) *Sensitivity of the Eye to Energy of Different Wave-lengths.* By examining a large number of persons, Nutting* has obtained data connecting the wave-lengths of radiant energy and the luminous sensation produced per unit of power in that wave-length.

He expresses his results in the form

$$V_\lambda = V_m \left(\frac{\lambda m}{\lambda}\right)^a e^{a\left(1-\frac{\lambda m}{\lambda}\right)} \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

where V_m is the photometric value of a unit of power in the wave-length of maximum sensitivity λ_m, a is a constant, e is the base of Napierian logarithms.

For the luminous intensities ordinarily employed in photometry he gives $a=181$ and $\lambda_m=0{\cdot}55\mu$.

Combining the expression for power distribution and the sensitivity curve for the eye (equations 2 and 3), we obtain for the photometric value of radiant power of wave-length λ, $E_\lambda V$, and for the photometric value of the whole of the radiant power

$$\int_0^\infty E_\lambda V_\lambda d\lambda. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

* "Bulletin" of the Bureau of Standards, Vol. V., p. 261, and Vol. VI., p. 337.

If the power distribution can be represented by

$$E_\lambda = P\lambda^{-n} e^{-\frac{Q}{\lambda T}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

Nutting* shows that the above expression for the photometric value of radiant power reduces to

$$L = A\left(1+\frac{B}{T}\right)^{-\rho} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

where the photometric value of the total radiant power in lumens at a given temperature, T, is represented by L,

$$A = PV_m \lambda^a{}_m e^a (a\lambda_m)^{-\rho} \Gamma\rho.,$$

$$B = \frac{Q}{a\lambda_m},$$

$$\rho = n+a-1.$$

It follows from what has been said above that the lumens radiated by carbon or tungsten filaments should be capable of being represented by an expression of the form of equation (6). Combining with this expression that for the watt-temperature relationship for the filament under consideration (equation 1), we get for the equation connecting lumens per watt and temperature

$$\frac{L}{W} = A_1\left(1+\frac{B}{T}\right)^{-\rho} T^{-m},$$

or expressed for convenience in the logarithmic form

$$\text{Log}\,\frac{L}{W} = C - m \log T - \rho \log\left(1+\frac{B}{T}\right). \quad . \quad . \quad . \quad . \quad (7)$$

From the measured values of lumens/watt and temperature, which are plotted in Fig. 3, the following values of the constants in the foregoing equation are found.

For carbon filaments

$$\text{Log}_{10}\,\frac{L}{W} = 21{\cdot}51 - 4{\cdot}58 \log_{10} T - 185 \log_{10}\left(1+\frac{155}{T}\right) \quad (8)$$

and for tungsten filaments

$$\text{Log}_{10}\,\frac{L}{W} = 23{\cdot}31_2 - 5{\cdot}1 \log_{10} T - 185 \log_{10}\left(1+\frac{155}{T}\right). \quad (9)$$

* *Loc. cit.*

The curves drawn in Fig. 3 are those derived from these equations, and it will be seen at once how nearly the observations fall on the curves; in fact, it would hardly be possible to find a form of curve which would fit the observations better. The error in temperature rarely exceeds 2 per cent. and in most cases is considerably less than 1 per cent.—*i.e.*, within the possible error of the experiments. Further, it is shown later that a very large extrapolation of the tungsten curve by this formula indicates a value for the melting point of tungsten which is not inconsistent with that found by other observers. The formulæ indicate that the maximum attainable efficiency would occur in the region of 6,000°C., which is quite in accord with accepted theories.

The origin of the constants in equations 7, 8 and 9 should be particularly noted. The watts-temperature relationship for a lamp has been found to be of the form $W \propto T^m$ (equation 1), m being a constant which appears in equation (7). The constant ρ of equation (7) is equal to $n+a-1$, where "$-n$" is the index of λ in equation (2) of the "Wien" form, which is assumed to give the power distribution curve for the filament throughout the visible part of the spectrum, and where "a" is derived from Nutting's equation for the sensitivity of the eye and has the value of 181. $B=\frac{Q}{a\lambda_m}$ (*see* equation 6), where Q is the other constant in the assumed Wein equation for power distribution, "a" has the value as before of 181, and λ_m is the wave-length of the energy to which the eye is most sensitive, *i.e.*, 0·55μ. Hence $B=\frac{Q}{99{\cdot}55}$. For a true black or grey body $m=4$, $n=5$, $\rho=185=14{,}500$, and B=145·0 approximately.

Before leaving the consideration of these equations connecting lumens per watt and temperature, it is desirable to discuss one or two points which at first sight may appear to have an important bearing on the deductions that can be made from the foregoing results.

Firstly as regards "n" in equation (5) and "m" in equation (7). For a black body the value of "n" in equation (5) is 5, and it will also be 5 for a true grey body whose radiation in all wave-lengths bears a definite proportion to that of a black body. It will not necessarily be 5, however, for selective bodies, although, as in the selective bodies under consideration,

they appear to radiate very much like grey bodies over *the visible spectrum.*

In equation (7) the constant "m," which is derived directly from equation (1), can only be regarded as connected with "n" ($n=m+1$) in equation (5) if the latter represents the distribution of power throughout the *whole* spectrum, and not merely in the visible spectrum. This latter is the assumption made in using equation (5) in this investigation, and the extent of the work described here does not justify the wider application of equation (5) to the whole spectrum for substances which do not behave as true black or grey bodies.

Lummer and Pringsheim*, investigating platinum, state that the distribution of power throughout the whole spectrum for platinum is given by the following equation :—

$$E_\lambda = a\lambda^{-6} e^{-\frac{15,600}{\lambda T}} \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

corresponding to the form for an ideal black body of

$$E_\lambda = C_1\lambda^{-5} e^{-\frac{14,500}{\lambda T}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

If this assumption were justifiable, the authors values for tungsten work out very nearly the same as those given by Lummer and Pringsheim for platinum ; but, for the reasons just stated, too much significance must not be attached to this agreement.

Coblentz† expresses the opinion, based on an investigation of several metals, that "n" in equation (5) is not a constant, but is a function of wave-length and temperature.

There is nothing in the results discussed above which is inconsistent with either Lummer and Pringsheim's or Coblentz's suggestions.

Throughout this work the practical case has been dealt with of filaments mounted in exhausted globes in which there is undoubtedly some loss of watts and efficiency due to the cooling effect of the leading-in wires. If conclusions of a fundamental nature are to be drawn from the results obtained, it is necessary to know to what extent this cooling is likely to affect the constants given in equations (8) and (9).

In the work already referred to by Hyde and Cady, figures are given for the loss of watts by conduction at the ends of

* Lummer and Pringsheim, "Verhandlungen der Deutschen Phys. Gesell.,' pp. 23-25, 1899.

† "Bulletin" B.S., Vol. V., pp. 338-379, 1908-1909.

filaments, and columns I. to IV. of the following Table are from this Paper.

TABLE III.

I. Lamp.	II. Watts / mean horizontal candles.	III. Watt loss.	IV. Efficiency loss.	V. Lumens / Watt.	VI. Temp. ° Abs.
Carbon 115 volts ...	3·1	2%	4%	$3{\cdot}4_5$	2,085
" " ...	1·8·0	3%	5%	$0{\cdot}5_9$	1,730
Tungsten 115 volts, 60 watts	1·25	4%	7%	8·0	2,280
" "	$11{\cdot}_0$	8%	16%	$0{\cdot}9_1$	1,745

These losses are calculated for different efficiencies as a percentage of the watts which would be required to maintain the filament throughout its whole length at the temperature of its midpoint, assuming no loss by conduction.

In columns V. and VI. are tabulated the values of lumens per watt calculated from Column II. using average values for the reduction factors of the types of lamp under consideration, and the corresponding temperatures taken from the curves in Fig. 3.

The difference between the temperature corresponding to the colour of the light radiated from the *centre* of the filament of any lamp of the above types, and that radiated from the *whole* filament are given in Table II. The following results are obtained by using the values in Tables II. and III. for ascertaining what would be the behaviour of the filaments used in this investigation had there been no cooling.

Carbon Lamps.

In ordinary lamps the watts and temperature are connected by the relation

$$\text{Log}_{10}\text{W}=\text{C}_1+4{\cdot}58\ \text{log}_{10}\text{T}\ .\ .\ .\ .\ .\ .\ (12)$$

Allowing for the watt loss as per column III., Table III., due to conduction, the watt-temperature relation for a filament kept at uniform temperature throughout its length and having no conduction losses is :

$$\text{Log}_{10}\text{W}_1=\text{C}_1-0{\cdot}191+4{\cdot}63_5\ \text{log}_{10}\text{T}\ .\ .\ .\ .\ (13)$$

Likewise for ordinary lamps the relation between lumens per watt and temperature is expressed by

$$\text{Log}_{10}\frac{\text{L}}{\text{W}}=\text{C}_2-4{\cdot}58\ \text{log}_{10}\text{T}-185\ \text{log}_{10}\left(1+\frac{155}{\text{T}}\right).\ .\ (14)$$

Allowing in a similar manner for the efficiency losses (column IV., Table III.), the lumens per watt and temperature relation for the ideal filament is given by

$$\text{Log}\frac{L}{W}=C_2+0{\cdot}199-4{\cdot}63_5\log T-185\log\left(1+\frac{155}{T}\right). \quad (15)$$

Tungsten Lamps.

Similarly the equations for tungsten filaments are changed when cooling is allowed for, from

$$\text{and}\quad \text{Log}\,W=C_3+5{\cdot}1\log T, \quad (23)$$

$$\text{Log}\frac{L}{W}=C_4-5{\cdot}1\log T-185\log\left(1+\frac{155}{T}\right) \quad (24)$$

to

$$\text{and}\quad \text{Log}\,W=C_3-0{\cdot}700+5{\cdot}3\log T, \quad (25)$$

$$\text{Log}\frac{L}{W}=C_4+0{\cdot}703-5{\cdot}3\log T-185\log\left(1+\frac{155{\cdot}4}{T}\right) \quad (26)$$

and the same phenomenon is observable as for the carbon filament lamps.

By comparing equations (13) and (15) and (25) and (26) respectively and remembering that (13) and (25) deal only with the *watt* relationship, omitting lumens altogether, it will be seen that practically all the change produced by cooling is in the watts. That means that the loss of lumens at the ends of a filament is virtually equal to the gain in lumens due to the centre of the filament (as shown in Table II.) being at a slightly higher temperature than that ascribed by the colour identity method to the filament as a whole.

The fact that temperatures based on the colour identity method and covering so wide a range are thus found to fit in so well with fundamental theory seems to afford presumptive evidence that the colour identity method gives values of temperature which are accurate at least from the relative point of view, and affords further experimental support for the conclusions which are arrived at on page 233 as a result of a general consideration of the phenomena underlying colour identity.

3. *Temperature of Lamp Filaments and of Melting Platinum.*

The foregoing experiments assume that a temperature ascribed to a filament by comparing the hue of its total radiation with that of a black body of known temperature approxi-

mates to the true temperature of the filament although the latter is glowing under open radiation conditions. If this approximation can be shown to be a very close one, the method might be of considerable use in certain branches of practical pyrometry. In what follows enough evidence is given of the correctness of the temperatures determined by the colour identity method, to justify the assumption for certain substances and to warrant a more complete investigation of the subject.

As regards the determination of filament temperature by previous observers, only three sets of determinations, those of Forsythe,* Von Pirani and Meyer† and Langmuir‡ are given in comparable form.

In some other determinations no mention is made of the watts per candle or lumens per watt of the lamps tested. In others "black body" temperatures and not true temperatures are given of the filaments radiating in the open.§

In Table IV. the results obtained by the authors are compared with those of the above-mentioned observers. Both Forsythe and Von Pirani only measured the mean horizontal candle-power of their lamps, and a reduction factor of 0·85 for carbon and 0·79 for tungsten has been assumed in both cases for the ratio $\frac{\text{M.S.C.P.}}{\text{M.H.C.P.}}$ It will be seen that the authors' results agree very closely with those of Forsythe, and it should further be noted that whilst agreeing with Forsythe, who used the usual optical methods, in the case of comparatively non-selective carbon filaments, they also virtually agree with his results for tungsten, although the latter is admittedly selective.

Von Pirani and Meyer found values of true temperature which are appreciably higher than Forsythe's and, therefore, also higher than by the colour identity method given here. They are given in column IV. of Table IV., the values being taken off a curve through Von Pirani and Meyer's values and

* "Phys. Rev." Vol. XXXIV., May, 1912.

† "E.T.Z.," 1912, May 2, p. 457 and July 11, p. 725.

‡ "Proc." of Amer. Inst. of Elect. Engineers. Vol. XXXII., p. 1895.

§ Dr. H. Lux, "E.T.Z.," May 28, 1914, gives tables connecting temperature and watts per mean spherical candle of tungsten lamps. The values given in the table would appear to approximate to true temperatures, but the method described for determining the temperatures is that ordinarily used for obtaining black body temperature. Without further information of the methods used by Dr. Lux for the determination of true temperature, a useful comparison with readings of other observers is difficult to make.

reduced to the same basis of lumens per watt. A considerable amount of this difference appears to be accounted for by the use by Von Pirani of a temperature scale which gives the melting point of platinum at 1,790°C.

TABLE IV.

Type of lamp.	Lumens per watt.	True Temperature of Filament °C.		
		Forsythe.	Pirani and Meyer.	The Authors.
Tungsten	$8{\cdot}0_5$	1,980	2,069	2,010
,, ...	$8{\cdot}1_3$	1,982	2,072	2,014
,, ...	$8{\cdot}4_4$	2,008	2,084	2,027
,, ...	$8{\cdot}5_3$	2,020	2,100	2,041
,, ...	$9{\cdot}0_6$	2,025	2,109	2,051
,, ...	$8{\cdot}8_6$	2,035	2,101	2,044
,, ...	$9{\cdot}3_6$	2,040	2,121	2,063
Carbon	$3{\cdot}5_2$	1,820	1,935	1,818
,, ...	$3{\cdot}9_5$	1,847	1,966	1,846
,, ...	$3{\cdot}9_4$	1,843	1,965	1,845

The lumens per watt are obtained from the values given by the authors of watts per candle, by assuming ratios of 0·79 and 0·85 respectively for the reduction factors $\left(\frac{\text{M.S.C.P.}}{\text{M.H.C.P.}}\right)$ for tungsten and carbon lamps.

The value of 0·9 is taken for the ratio of the Hefner to the British units of candle-power.

Both Von Pirani and Forsythe determined black body temperatures and added an amount depending on certain assumptions in order to get true temperatures. It is not possible to correct Von Pirani and Meyer's figures to make them comparable with those based on a temperature scale which gives the more usually accepted melting point for platinum, but it is clear that if this could be done Von Pirani and Meyer's figures would agree much more closely with Forsythe's. Langmuir does not give details of how he obtained his temperature values, since his Paper was not directly concerned with the measurement of temperature. His results differ by about 2 per cent. from the authors' values obtained by extrapolating the curve shown in Fig. 3 of this Paper, using the formula given on page 246.

In order to ascertain if the colour identity method is correct for substances other than carbon it is necessary to know the true temperature of some glowing filament. The melting point of platinum is now very generally accepted as (1750±20)°C.* If a filament of platinum could be gradually raised to the melting point by an electric current and compared at its

* Burgess-Le Chatelier. "Measurement of High Temperatures," p. 492.

melting point against one of the colour standards, the "colour identity" temperature of the platinum could be fixed at the melting point and compared with the known melting point of platinum. Platinum is admittedly a selective body. In addition, when radiating in the open its black body temperature determined by the ordinary optical methods (λ=0·650μ) is some 200°C. lower than the true temperature, and as the colour comparison would be made under open radiation conditions, the experiment should be a crucial one for proving if the colour identity method gives true temperatures.

The only precaution necessary is to take a fair length of platinum wire and use only the central portion so that the cooling of the ends of the wire by the leading-in terminals shall not influence the determination. Lengths of No. 25 gauge wire were used, 13 cm. long, of which all but the centre 5 cm. was screened off. Simultaneous comparisons were made with both carbon and tungsten colour standards. The current in the platinum wire was slowly raised and that in the colour standards increased so that identity of colour was always maintained in the photometer up to the melting point of the platinum. The colour standards were arranged each with a photometer head on either side of the platinum wire, and no difficulty was found in maintaining colour identity to the point at which the platinum melted.

Table V. gives the results of all the 15 determinations.

TABLE V.

Experiment.	Temperature of platinum at melting point by colour identity method °C.	
	With carbon filament colour standard.	With tungsten filament colour standard.
1	1,752°C.	1,765°C.
2	1,746°C.	1,770°C.
3	1,727°C.	1,751°C.
4	1,737°C.	1,765°C.
5	—	1,747°C.
6	1,761°C.	1,784°C.
7	1,769°C.	1,779°C.
8	1,755°C.	1,784°C.
9	1,763°C.	1,782°C.
10	1,759°C.	1,782°C.
11	1,727°C.	1,765°C.
12	1,737°C.	1,769°C.
13	1,747°C.	1,782°C.
14	1,764°C.	1,789°C.
15	1,757°C.	1,789°C.
Mean	1,750°C.	1,773°C.

The mean result gives the melting point of platinum as 1,750°C. by the carbon filament lamp and 1,770°C. by the Tungsten lamp, a result so near to the accepted value of 1,750°C.* as to afford strong evidence of the reliability of the colour identity method. It is intended later to repeat this experiment with other metals such as nickel, iridium or rhodium, using a neutral atmosphere to surround the incandescent wires. The difference of 1 per cent. in the temperatures given respectively by the carbon and the tungsten colour standards must not be assigned too much weight. Although the method is capable of a greater accuracy than this, it is not claimed that the determinations described here are correct to 1 per cent.

Comparing again the usual optical methods and the colour identity method of estimating temperature, it is worth while to see what is the explanation of the phenomenon which has been described in this Paper. The factor of chief interest is, that if a black body at 1,750°C. radiates towards one side of a photometer and platinum at the melting point (1,750°C.) towards the other, there will be identity of hue on the two sides of the photometer, even though the platinum is operating under open radiation conditions. The hue of the total radiation of the platinum is, therefore, a measure of its true temperature.

If, on the other hand, a pyrometer be used to measure the temperature, first, of the black body and then of the melting platinum, it will give a value of 1,750°C. for the black body and about 1,550°C. for the platinum. The latter temperature will depend on the wave-length in which the measurements are made, but whatever the wave-length used the temperature given will be very much lower than the true temperature of the platinum.

Waidner and Burgess have given the melting point temperatures of platinum (black body) determined in three wave-lengths as follows :—

TABLE VI.

Colour.	Wave-length.	Melting point.
Red	0·666μ	1,534°C.
Green	0·547μ	1,578°C.
Blue	0·462μ	1,610°C.

Examining the cases of a grey body and then of platinum, for which most data are available, it is possible to see from what

* Burgess-Le Chatelier, "Measurement of High Temperature," p. 492.

follows how closely the colour identity method will tend in practice to give the true temperature.

Curve A (Fig. 5*) shows the power distribution of a black body at 1,750°C. over the visible spectrum calculated from the Wien equation. Curve C is a curve for a grey body with a certain emissivity at the same *true* temperature. If the temperature of this grey body be measured with an optical pyro-

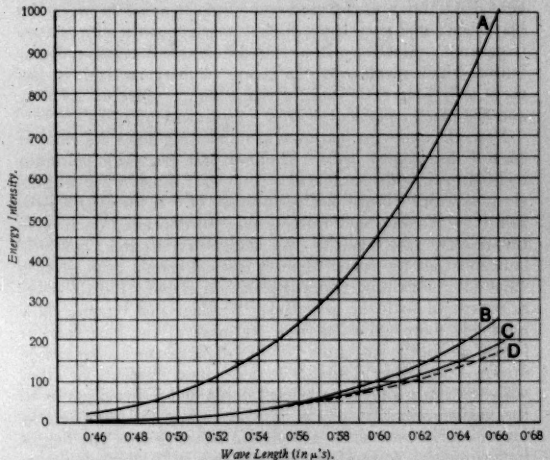


FIG. 5.

Energy intensity curves in the visible spectrum for :—
A. Black body at 1,750°C.
B. „ „ at 1,578°C.
C. A grey „ at 1,750°C.
D. Platinum under open radiation conditions plotted from values given by Waidner and Burgess.

meter, in the green (λ=0·547μ) it will be given a value of 1,578°C. This particular grey body curve has been taken in order that it may be comparable with Waidner and Burgess's platinum curve, in which the black body temperature of the platinum melting point at λ=0·547μ, is given as 1,578°C. Curve B is the true black body curve for a temperature of

* Hyde, "Journ." Franklin Inst., *loc. cit.*, shows curves very similar to these to illustrate points connected with his investigations on selectivity.

1,578°C., and has been drawn in order to show the difference between the black body and grey body power curves referred to the same intensity of radiation in the green.

The ordinates are relatively so small at the shorter wave-lengths that the crossing of curves B and C at $\lambda=0{\cdot}547\mu$ cannot be distinguished. They do, however, actually cross. It is well to notice from curves B and C what a difference in light distribution at different wave lengths exists between the black body and grey body at the same apparent temperature (measured optically). It is this difference of relative distribution which results, by the colour identity method, in the grey body (curve C) being given a temperature of 1,750°C. and the black body (curve B) a temperature of 1,578°C.

Now, the curve for platinum at its melting point lies close to curve C for a grey body. Waidner and Burgess's platinum melting point determinations for the three wave-lengths given in Table VI. are plotted in curve D. If the figures published by Waidner and Burgess may be depended on to give the relative intensities in the three wave-lengths, curve D indicates that the colour identity method should have given a value for the melting point of platinum above 1,750°C., since, as compared with the grey body, Waidner and Burgess show relatively more radiation from platinum at the blue than at the red end of the spectrum. Before definite conclusions can be drawn it would be desirable to have measurements in other than the three wave-lengths considered and information as to the monochromatism of the light in each of the wave-lengths for which the intensities are plotted in curve D. The figures for the melting point of platinum, shown in Table V., indicate, it is true, a tendency to fall in the direction to be expected from Waidner and Burgess's values plotted in the diagram, but not, however, as much as line D indicates. The opinion, therefore, expressed by Hyde that the colour identity method will err in ascribing temperatures which are, if anything, slightly too high, is supported so long as the bodies in question are selective *in the visible spectrum* in favour of the shorter wave-lengths. If they are selective in favour of the red end, the temperature ascribed will tend to be low whilst if they are true grey bodies the method will be accurate.

It should, however, be recalled that compared with other substances platinum is regarded as a relatively selective body in the visible spectrum, and if this is so, the differences indicated in Fig. 6 are for a fairly extreme case,

Little is actually known as yet regarding the departure of metallic bodies from the characteristics of grey bodies in the *region of the visible spectrum*. Any deviation which there is would seem to be small in amount and insufficient to invalidate estimations of the temperature by the colour identity method, intelligently used. It is suggested that the method should be specially useful for assisting in the determination of temperatures and melting points of some of the more refractory substances whose true temperatures by the usual optical methods and assumptions are admittedly open to doubt.

It is of interest now to see what are the temperatures of filaments in gas-filled lamps determined by the colour identity method of measurement.*

In estimating the temperature of the filament in a lamp bulb containing gas it must be remembered that the relation between lumens/watt and temperature cannot be the same as in the ordinary vacuum lamp because of the considerable number of watts carried away from the filament by convection in the gas. Further, the proportion of watts convected depends considerably on the diameter of the wire and on the density of the gas in the bulb.

Six gas-filled lamps have been compared against an ordinary vacuum tungsten lamp in order to determine the difference of efficiency expressed in watts per mean spherical candle or lumens per watt between the two types of lamps when the colour of their radiations is identical, and therefore, when, to a close approximation their temperatures are the same.

Comparisons were made up to an efficiency of about 0·75 watts/mean spherical candle for the vacuum lamp, or about 0·9 watts/mean spherical candle for the gas-filled lamp. The results are shown in Fig. 5, where watts/mean spherical candle for the vacuum lamp is plotted as ordinate and watts/mean spherical candle for the gas-filled lamp as abscissa. The considerable difference between these curves must be ascribed to differences in the amount of gas in the bulbs and to varying diameters of filaments and spirals.

The curves have been extrapolated to pass through the zero of the diagram, and it will be seen that all the points lie on a straight line which passes through the origin except at the comparatively low values of efficiency.

* *See* Langmuir, "Proc." Amer. Inst. of Elect. Eng., Vol. XXXII., p. 1895.

The ordinary working efficiency of gas-filled lamps is at the present time about 0·7 watts/mean spherical candle, corresponding in identity of colour of radiation with the vacuum lamp at 0·5 watts/mean spherical candle—*i.e.*, 25 lumens per watt approximately.

From the equation to the curve shown on Fig. 3 this gives a temperature of 2,800°C. abs. for the ordinary working temperature of tungsten in gas-filled lamps.

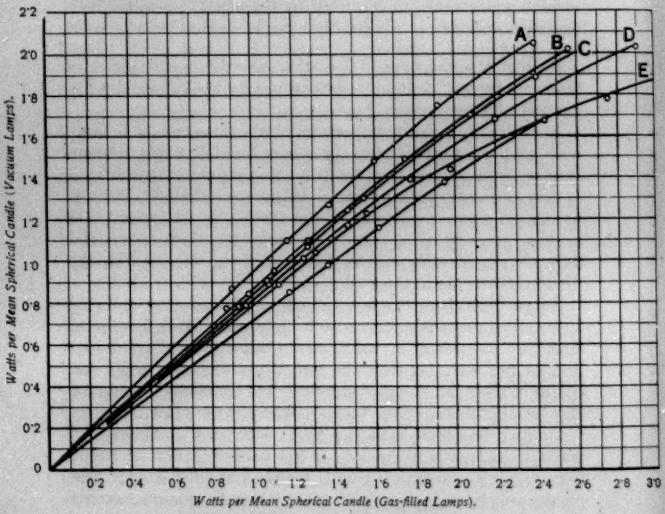


FIG. 6.—GAS-FILLED LAMPS.

Curves showing for various gas-filled tungsten filament lamps A to E, the relation between the actual watts per mean spherical candle and the watts per mean spherical candle of a vacuum lamp filament at the same temperature.

If the half-watt lamps are overrun they will be found to burn for a short time satisfactorily at 0·40 watts/mean spherical candle, corresponding to a vacuum lamp at 0·32 watts/mean spherical candle, or approximately 40 lumens/watt. Using the same formula this value of lumens/watt corresponds with a temperature of 3,100°C. abs. (2,830°C.). It is clear, then,

that the melting point of tungsten would be above this value, but it cannot be said with certainty how much higher. The comparison lamps used in the determination were of the vacuum type and themselves had tungsten filaments. At 3,000°C. abs. the blackening of the bulbs of such lamps is rapid and liable to lead to error if the temperature of the filament is pushed up to the melting point. The determination of the melting point of tungsten, therefore, by this method could only be undertaken if precautions were taken against errors due to blackening of the bulb. The experiment will be more successful when gas-filled colour standards are made by comparison direct against the black body. Burgess and Le Chatelier* give the melting point of tungsten as $(3{,}000 \pm 100)$°C., a value with which the authors' observations by the colour identity method are not inconsistent at the highest temperature at which they could safely make measurements.

4. *The Colour of Radiation from the Violle Standard.*

At the beginning of this Paper the importance was explained of giving proper consideration to the colour of the radiation from any proposed standard of light. The increasing efficiencies of modern sources of light owe these increases mainly to higher running temperatures, and hence the light emitted tends to consist of a relatively larger proportion of shorter waves. The light from the Hefner lamp has been found by the authors to correspond in colour with that from a black body at 1,540°C. That from the Pentane lamp to 1,610°C.

The Violle standard consisting as it does of platinum at the melting point (1,750°C.) has a radiation which corresponds with the colour of a carbon filament glow lamp running at an efficiency of $2{\cdot}6_5$ lumens per watt, or $4{\cdot}7_5$ watts per mean spherical candle (*see* Fig. 3). Assuming a reduction factor of 0·85 this is equivalent to an ordinary carbon filament lamp with a specific consumption of 3·8 watts per mean horizontal candle.

The authors wish to express their acknowledgment in connection with this work to Dr. R. T. Glazebrook, C.B., F.R.S., Director of the National Physical Laboratory, and also to Dr. J. A. Harker, F.R.S., for facilities afforded in the use of the black body furnace referred to at the beginning of the Paper.

* *Loc. cit.*, p. 492.

DISCUSSION.

Prof. S. P. Thompson thought something must be wrong with the English language when "white" light had to be defined as that radiated by a "black" body. The term "grey" body seemed indefinite. There were many shades of grey, and it would be interesting to know how many bodies were really "grey" according to the precise definition of the authors. He thought that in plotting lumens per watt against temperature it would have accorded better with custom to have plotted temperatures as abscissæ instead of as ordinates.

Mr. A. P. Trotter (communicated remarks): If this Paper had preceded the introduction of the optical pyrometer, that instrument would not perhaps have gained such a footing. Notwithstanding the apparent advantage that the intensity of light for any given wave-length varied as the fifth power of the temperature, while that of the whole light varied as the fourth power, the principle of colour identity seemed clearly to be the right one for measuring high temperatures, and the use of an arbitrary coloured screen, the wrong one. The estimation of high temperatures by observing the colour of an incandescent body was perhaps practised by Tubal Cain in judging forging and welding heat. What the blacksmith calls "just red at the back of the forge" is about 400°C., and "cherry red" is about 1,650°C., but it has remained for the authors to convert it from a matter of judgment to one of accurate measurement. The method directs attention to small colour differences which in ordinary photometry are deliberately and sometimes with difficulty ignored. This difficulty drives some observers to the flicker photometer. The difficulty is reduced by experience. Possibly some eyes are more sensitive to differences of brilliance and others to differences of hue. The authors are justified in the use of the expression "grey body." The ideal black body which absorbs all radiations and reflects none, and for any temperature radiates more than any other body, is rather repulsive to some people, for it is inseparably connected in their minds with the complicated laws of Stefan, Planck and Wien. The opposite of this would be a body which absorbs no radiations, is a perfect reflector, does not emit any radiation when at a high temperature and entails no mathematics. No such body is known. Intermediate bodies are of two classes, and in their radiant properties they may be called selective and grey, just as bodies between the extreme hues of black and white are coloured and grey. There is an indefinitely large number, several thousands, of perceptibly different shades of grey, and the degree of greyness of a radiant body is merely the factor by which the ordinates of such a curve as C, Fig. 5, must be multiplied to fit curve A. If it does not fit, but cuts it, it is not grey but selective. The authors do not seem to be justified in stating that white light means light of any colour emitted by a black body. It is generally recognised that there can be no definite standard of white light, but there are many substances such as magnesia, snow, or even paper, which are white, and white light is best represented by sunlight reflected from such a substance. But the convention is that it must be our sun, not the hotter Sirius, or the cooler Antares. A grey body also reflects pure white light, but not so much as a white body. In colorimetric tests as made in chemical and physiological laboratories it is found that two tints may appear to one observer to be of identical hue, while to another they differ. This occurs when the two hues differ spectroscopically, and when the eyes of the observers differ in colour sensitivity. In the present method of colour identity, so long as the hues are in general of similar spectroscopic character this difficulty would not arise. When a piece of white porcelain with a dark pattern is heated red hot, the pattern becomes brighter than the background. An optical pyrometer would indicate that the dark pattern was at a higher temperature. But the author's method would reduce the brilliance of the pattern to that of the ground, would show that the hues are identical, and would indicate equality of temperature. The weak point of the ordinary optical pyrometer with a screen as monochromatic as is practicable is that it is applicable only to bodies which approxi-

mate to the conditions of the ideal black body. It is interesting that so useful a side issue should have arisen from the investigation of the Violle standard of light which seems less likely to become a practical one since the publication of Dr. Petavel's work than it did before.

Dr. J. A. Harker said that to one accustomed to look at the physics of optical pyrometry in the ordinary way it is difficult at first sight to see why platinum at its melting point should emit the same colour of radiation as a black body at the same temperature. In fact one's predisposition would be to the opinion that this is quite unlikely. The accuracy with which the authors by their method determined the value for the melting point of platinum was very surprising, and the considerations the authors bring forward put a new complexion on one's conceptions. With regard to the "black body" used by the authors, it was extremely difficult to obtain a furnace at these high temperatures without a cloudy atmosphere, but after distilling the impurities out of the carbon—which was the only suitable substance to use—he had found it possible to obtain a high temperature furnace with a perfectly clear atmosphere. With the ordinary optical pyrometer it was impossible by means of the coloured glass supplied to get sufficiently perfect monochromatism to give great accuracy. If a strip of platinum be used instead of a wire it is possible to maintain it within a degree or two of its melting point for some time.

Mr. A. Campbell asked how the optical pyrometer was calibrated at high temperatures. In the case of the Violle standard, was it essential actually to melt the standard strip? Would it not be more satisfactory to use a tungsten strip heated up until a small speck of (*e.g.*), quartz on it began to melt rather than to melt the strip which was under observation?

Mr. J. S. Dow doubted if metal filament lamps could be regarded as strictly grey through their visible spectrum. He recalled a series of articles by W. Coblentz in the "Illuminating Engineer," in 1910, showing how metals in general had a low emissivity in the infra-red, and this suggests irregularity in the emissivity of a polished metal surface, even in the visible spectrum. The methods the authors proposed seemed to require less manipulative skill than most pyrometers. For example, one observer had discovered an interference effect that might give rise to considerable errors in the case of instruments involving the inspection of a bright filament against a luminous background. Measurements with a photometer would be free from this source of error. The authors had found that half-watt lamps could be run for a short time at 0·4 watt per mean spherical candle. Could they, by extrapolation, state the approximate limiting theoretical efficiency of a tungsten filament? Dr. Lux had recently estimated that the melting point of tungsten would be approached at 0·3 watt per candle.

Mr. A. W. Beuttell suggested that any difference in the colour sensations of the observer from the normal would affect the results obtained by the method.

Dr. C. Chree asked which of the values, 1,750°C. or 1,770°C., the authors considered most nearly correct.

Mr. E. H. Rayner thought it might add to the sensitiveness of the method if, when the colour match had been obtained, coloured glasses—say, first a red and then a blue glass—were put in front of the eye. Any inexactness in the match might be increased and shown up in this way.

Mr. J. Guild (communicated remarks): The auxiliary adjustment for equality of brightness mentioned by the authors on page 236 is an important one, inasmuch as, on account of the change in the sensitivity curve of the eye with change in brightness of the incident light, it is only when both sources produce the same illumination *at the eye* that identity in the colour perceived involves identity in their energy distribution curves over the visible spectrum. If, for example, one adjusted two sources to give identity of hue but neglected to equalise the brightness at the photometer the source producing the weaker light would require to be at a lower temperature, *i.e.*, to have more red in its spectrum than the other, since, for the weaker light,

the red-sensitivity of the eye bears a smaller ratio to the total sensitivity than for the stronger. In what way does the accuracy of the colour-identity method vary with temperature? To the casual observer the colour of a furnace appears to vary much more rapidly with temperature at low red and orange heats than at higher temperatures, and one would expect considerable precision at, say, 700°C. or 800°C., and a progressive falling off of sensitiveness as the temperature is increased. This could be calculated from the radiation laws and the chromatic properties of the eye, but a series of test experiments over a wide temperature range would be more convincing.

Mr. Paterson thanked the speakers for their remarks. In answer to Prof. Thompson, Mr. Trotter's contention regarding "grey" bodies seems to explain the matter clearly. The convention of restricting the term "grey" body to one which radiated in the way shown in curve C, Fig. 5, is a very usual one. There could be an indefinite number of such curves, but each had the property that some one multiplier would make all its ordinates coincide with a black body curve at the same true temperature as the grey bodies in question. A body whose curve would not conform with this requirement would be spoken of as a "selective" body rather than as a "grey" body. The question is mainly one of nomenclature and definition. In the American literature the above convention is generally accepted. According to Prof. Thompson and Mr. Trotter the authors may be wrong in suggesting that white light is the light emitted by a black body at any temperature. Mr. Trotter prefers to restrict it to light radiated by the sun. He was still of opinion that "white light" was a good term to use, since it conveys the impression of a mixture of all wave-lengths in definite proportions. If "white light" were to be regarded as the light emitted by the sun, it would be very indefinite owing to atmospheric absorption.* In reply to Mr. Campbell, no effort has been made by the authors to use platinum strip as a standard of light, but experiments had been made by others on these lines without much success. In reply to Mr. Dow, several experimenters had come to the conclusion from the evidence at their disposal that these materials were mainly selective in favour of the visible spectrum as a whole. The authors had now shown experimentally that platinum had the same relative distribution of light as a black body at the same temperature, and this appears to the authors important evidence to justify the last statement, that is to say, that they approximate to "grey" bodies in the visible spectrum. The authors had compared a tungsten filament with a carbon filament over the visible spectrum with a spectrophotometer, and to the accuracy to which the instrument could be used no difference in selectivity could be detected. Over the infra-red portion of the spectrum it is well known that a large difference exists. An answer to the question regarding the limiting efficiency of half-watt lamps is given in the Paper. In reply to Mr. Beutell, a person with defective colour sight should only be less sensitive than one with normal vision, but on the average should not obtain different results. In reply to Dr. Chree, the melting point of platinum is usually given as 1,750°C. + or − 20°C. The present experiments were carried out not to determine the melting point of platinum but to show from the melting point values obtained the accuracy of the method of colour identity. Mr. Rayner's suggestion is certainly worth trying and ought to yield very interesting results. In reply to Mr. Guild, it is impossible to obtain identity of colour with accuracy unless illuminations at the two sides of the photometer are the same. There is a slight difference in sensitiveness of the colour identity method at different temperatures, for instance, a definite change of colour which is caused by 1 per cent. in temperature at 2,000 deg. abs. is caused by 1·6 per cent. at 2,800 deg. abs. Regarding Dr. Harker's remarks, the authors hope to take advantage of the offer made by him to use one of his black body furnaces up to higher temperatures. In doing so they would add to the obligation they are already under to Dr. Harker in connection with the furnaces used in the present work.

* *See* "Modern Illuminants." Gaster and Dow. P. 181.

XX.—*The Unit of Candle-power in White Light.** By Clifford C. Paterson *and* B. P. Dudding, *A.R.C.Sc.* (*From the National Physical Laboratory.*)

Synopsis.

1. The differences in the colour of the light radiated from different sources of white light causes uncertainties in photometric determinations. The Paper describes the methods used at the National Physical Laboratory to minimise this well-known difficulty by the use of the cascade principle.

2. The six sets of electric substandards of candle-power are described, which vary in the colour of the light radiated, from that of the pentane lamp (red) to that of a tungsten vacuum lamp operating at 1·5 watts per candle.

3. The differences between the values obtained by different observers in the process of stepping from one colour to the next are discussed and compared with those obtained by the direct comparison of the first and last sets of substandards.

4. The probable errors of the determination are discussed and shown to be of the order of $0{\cdot}2_5$ per cent.

5. The absolute value of the unit of candle-power has been re-determined from the pentane lamp, and found to agree with the determination made by one of the authors 10 years ago to within less than 0·1 per cent.

6. The corrections to the pentane lamp for humidity and barometric changes were also redetermined, and found to agree within narrow limits with those previously found by one of the authors.

7. The relation between the humidity and temperature corrections for the pentane lamp is discussed, and the fact that these two effects may act together is suggested as the reason for a discrepancy which has been noticed between the humidity constants of the lamp as determined in London and Washington.

* The work dealt with in the various sections of this Paper has entailed so many thousands of measurements and candle-power determinations of individual lamps, that to give them in detail would greatly overweight the Paper with tabular matter. The authors have, therefore, felt obliged to limit themselves to summaries of results and sometimes to bare statements of the results obtained, computed from their manuscript tables.

General Discussion.

The principal object of this Paper is to describe and discuss the methods which have been used at the National Physical Laboratory, and the results obtained in the standardisation of high efficiency metal filament lamps in terms of the unit given by the 10-candle pentane lamp. Such a standardisation entails the comparison of two sources whose radiation in the visible spectrum, though continuous, differs appreciably in hue. The amount of this difference may be gauged by remembering that if the voltage on an ordinary carbon filament lamp is lowered until the hue of its light matches that from the pentane lamp, the carbon lamp will be found to be burning at a rate of consumption of about 7·7 watts per mean horizontal candle. The ordinary consumption of carbon filament lamps is in the neighbourhood of 4 watts per mean horizontal candle, and of tungsten lamps between 1·0 and 1·5 watts per mean horizontal candle. If the Hefner lamp were matched on the same basis the specific consumption would be still higher, viz., at the rate of about 11·0 watts per candle.

In approaching this question some criterion is wanted in terms of which differences in the hue of light may be expressed. In this Paper, spectral colours are not being dealt with, but only so called white light—that is to say, light whose spectral distribution is continuous and not very different from that radiated by a black or grey body. The most common way of defining the hue of white light is to state the specific consumption in watts per candle of an incandescent lamp, the hue of whose radiation is identical with it. This value of specific consumption, however, will not be the same for a carbon as for a tungsten lamp, both of whose radiations have the same hue. A reason for the difference is the tendency for the tungsten lamp to radiate selectively in favour of those wave lengths lying within the visible spectrum. It is necessary, therefore, if this basis of comparison is used, to say if the watts per candle are on the carbon or tungsten basis, and hence it will be seen that although the values of watts per candle of incandescent lamps are convenient as a general guide to colour, this basis of measurement is arbitrary, and not very satisfactory for precise definition.

A series of comparisons was therefore carried out in order to express the hue of radiation from electric glow lamps in terms of the temperature of a black body whose radiation was identical in colour with that of the glow lamps in question. It

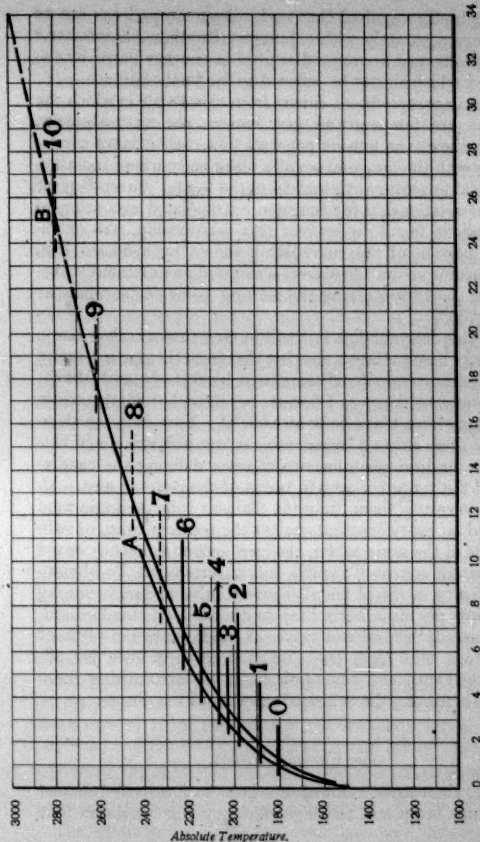


FIG. 1.—ORDINARY TUNGSTEN (CURVE B) AND CARBON (CURVE A) FILAMENT VACUUM LAMPS. CURVES CONNECTING LUMENS PER WATT* OF THE LAMPS WITH THE TEMPERATURE OF A BLACK BODY WHOSE RADIATION IS IDENTICAL IN COLOUR WITH THAT OF THE LAMPS.

* Assuming a ratio of $\frac{\text{M.S.C.P.}}{\text{M.H.C.P.}}$ of 0·85 and 0·79 for carbon and tungsten filaments respectively, the approximate watts per candle are obtained for any value of lumens per watt by dividing the latter into 10·7 for carbon filament lamps, and 10·0 for tungsten filament lamps.

has long been recognised that such colour comparisons can be made very accurately with a Lummer Brodhun photometer.* The methods used are those described in another Paper by the authors.† which should be referred to for further details.

In Fig. 1 are reproduced curves from this Paper in which the ordinates are temperatures and lumens per watt abscissæ. Curve A shows the lumens per watt of a carbon lamp corresponding with the temperature of a black body whose radiation is identical in colour with that from the lamp. Curve B gives the same relationship for tungsten extrapolated according to the equation found to express this relationship, the extrapolated portion of the curve being shown by a dotted line. Temperatures are on the optical scale, using a dominant wavelength of $\lambda=0{\cdot}650\mu$ giving the melting point of platinum as about 1,750°C.

From this diagram (line 1) it will be seen that a carbon lamp, the hue of whose radiation matches that from the pentane lamp, has the same coloured radiation as a black body at 1,610°C., whilst a tungsten lamp at 1·5 watts per mean horizontal candle (line 6) matches a black body at 1,950°C. Now, between these extremes there exists a considerable colour difference, and this has to be bridged photometrically when defining the candle-power of the tungsten lamp in terms of the pentane unit. If, in order to certify them, tungsten filament sub-standards were compared on every occasion against the pentane lamp, or sub-standards of the same colour, the large colour difference would be a constant source of trouble and uncertainty. To obviate this difficulty, a set of metal filament sub-standards running at 1·5 watts per mean horizontal candle has been standardised, by the cascade method and values of candle-power in terms of the pentane unit have been assigned to them once for all. The method used and results obtained in standardising these tungsten lamps are the principal subjects of this Paper.

The Cascade Method.

Briefly, the course followed has been to interpose between the pentane lamp and the high-efficiency sub-standards four

* Hyde, Cady and Middlekauff, "Selective Emission of Incandescent Lamps." Ill. Eng. Soc., New York. Vol. IV., 1909.

† Paterson and Dudding, "The Estimation of High Temperatures by the Method of Colour Identity." "Proc." Phys. Soc., London. Vol. XXVII., p. 230.

additional sets of lamps, each varying from the next in efficiency (or operating temperature) by such an amount that the difference in the hue of their radiations constitutes a regular colour gradation in approximately even steps from the pentane colour up to that of the 1·5 watts per candle tungsten filament standards. These steps are marked respectively by the horizontal lines 2, 3, 4 and 5 on Fig. 1.* If, now, each such set of sub-standards is standardised against the set below it—that is to say, the set whose light is one stage redder than its own—it is possible to arrive at a value for the final set by a series of photometric comparisons, in each of which the colour difference, although perceptible, is too small seriously to trouble an observer with normal sight.

It should be said at once that this system does not eliminate the personal error in colour photometry. It merely divides a large colour step into a number of small ones. It is known, however, that with scarcely perceptible colour differences the photometric agreement between different observers with normal sight is exceedingly close and the measurements can be made to a high accuracy. On the other hand, photometry with large colour differences, although possible in commercial work, gives rise to erratic and inconsistent readings when judged from the standpoint of the higher precision required for standardisation.

It is assumed in what follows that in the measurements of lights of different colours the true basis of comparison is the perception of the average normal eye. It is for the use of the human eye that artificial illumination is chiefly required, and it is suggested that two lights should be considered equally intense when the illuminations produced by them appear to be equal to a person with average normal sight.

The nearest approximation which is was possible to make to normal sight was to take the average of six different observers. Each one of these carried out the complete series of comparisons, and none of them had abnormal colour vision. The results of the investigation show that only very small errors can be introduced by assuming that the mean result will represent the average of a much larger number of persons.

The photometric work involved has been laborious. In order to eliminate any uncertainties of individual lamps and to

* Set 3 will be found to differ from its neighbours by smaller amounts than the other sets. It was included in the series because it corresponded in colour with the ordinary carbon filament standard lamps.

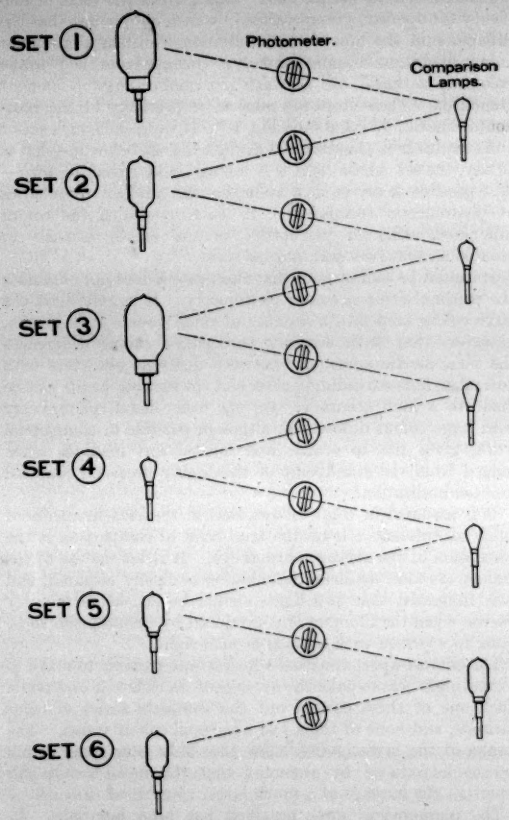


FIG. 2.—DIAGRAM SHOWING SCHEME OF "CASCADE" COMPARISON USED IN THE REALISATION OF THE UNIT OF CANDLE-POWER IN LAMPS OPERATING AT 1·5 WATTS PER MEAN HORIZONTAL CANDLE (SET 6).

reduce the probable error, each of the six sets of lamps consisted of 10 or 12 individual lamps, and each set was compared with the next one, in order of colour, some six or eight times by each observer. This entailed in all between 2,000 and 3,000 standardisations, or between 10,000 and 15,000 observations. The work was started several years ago, but the difficulty experienced at first in obtaining metal filament standards which were sufficiently constant caused much delay in the early stages.

Fig. 2 indicates graphically the scheme of the cascade comparison, and the relation of each group of standards to the remainder.

Set 1 represents a series of 11 sub-standards whose radiation matches in hue that of the pentane lamp, or the radiation from a black body at 1,610°C. Set 2 indicates a series of 11 Fleming-Ediswan sub-standards operating at an equivalent temperature of 1,700°C. Set 2 was compared against set 1 by the usual substitution method of photometry, using a Lummer-Brodhun photometer, and a comparison lamp, the colour of whose radiation lay midway between that of sets 1 and 2. In a similar way, set 3, with an equivalent temperature of about 1,750°C., was compared against 2 through another comparison lamp, which also divided the colour difference, and so on to set 6. It will thus be seen that by such a substitution method as this the colour difference between sets 1 and 6 is divided into 10 approximately equal steps—that is to say, at any one measurement there is only a colour difference of one-tenth that of the difference between sets 1 and 6. The difference of colour in the photometer throughout the investigation was, therefore, never greater than that resulting from a difference of temperature of about 45°C. All the measurements were made with an illumination at the photometer screen of 10 metre-candles. Finally, having by this means, and for each observer, assigned candle-power values to each of the lamps in set 6, the latter were compared in one step with set 1.

It is of interest to note that the colour of the radiation from melting platinum is the same as that of set 3, marked by line 3 in Fig. 1.*

Before passing to the details of the work the authors desire to point out one of the advantages of the cascade system as

* *Loc. cit.*

applied to electric sub-standards. The possession at a standardising laboratory of a regular gradation of sub-standards ranging from the lowest to the highest efficiencies, enables a set of standards of suitable colour to be chosen for whatever test has to be undertaken. As gas-filled lamps develop and come more into use there would seem to be no reason why the system should not be extended to very high efficiencies with sub-standards eventually covering a range of from 1,600°C. to about 2,700°C. The authors are at present experimenting with gas-filled standard lamps with the intention of carrying the work a stage further should they prove satisfactory.

It is found that four more steps are required to connect up the existing set 6, with the standards required for the testing of present-day gas-filled lamps. These are indicated by horizontal dotted lines in Fig. 1, marked 7, 8, 9 and 10.

Lamps Used in the Investigation.

The types of lamps used in the investigation are shown in Fig. 3. Some of them, such as the Fleming-Ediswan lamps (set 2) date back from 1902, and are the type of carbon filament standards due to Dr. J. A. Fleming, F.R.S., which have been used at the Laboratory since then for maintaining the unit of candle-power. Set 4 are " gem " lamps with so-called metallised carbon filaments, and were the best lamps which could be found at the time they were installed, for giving constant results at approximately 3·5 watts per mean horizontal candle. The remaining lamps have tungsten filaments and have been specially made for the Laboratory by the Osram Lamp Works and presented at different times by the General Electric Co. Reliable tungsten standards which can be depended on for constancy when operated at high efficiencies are not easy to obtain. Many tungsten lamps are liable to small erratic fluctuations of candle-power, the reason for which is obscure. Such fluctuations being small, do not affect the lamps for ordinary lighting purposes, but are a serious drawback for standard work. Each set of lamps, therefore, required careful preliminary testing, and many had to be discarded in the process. In the early days of the tungsten lamp several complete sets of standards had to be discarded after their seasoning was finished and standardisation work had begun, due to the appearance of small changes of candle-power.

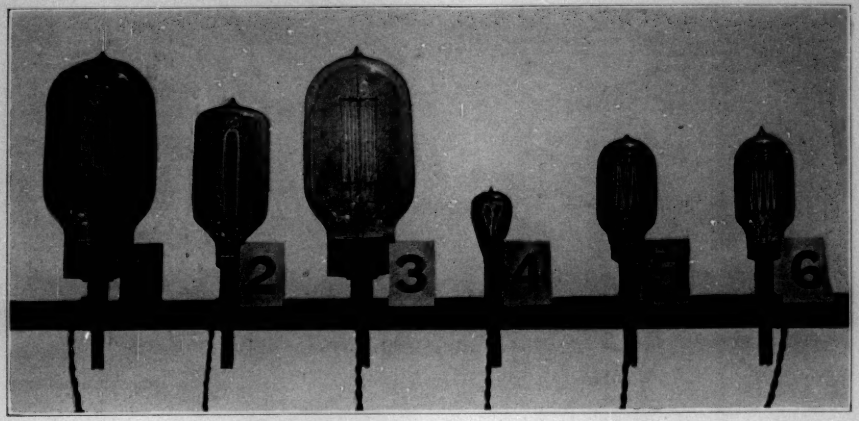


FIG. 3.—REPRESENTATIVE LAMPS FROM EACH SET OF SUB-STANDARDS.

To face page 270.

The details of the various sets of lamps are given in Table I.

TABLE I.

Set.	Approx. watts per M.H.C.	No. of lamps.	Temp. of black body for colour match.	Approx. c.p.	Description.
1	7·7 (Carbon basis)	11	1,610°C.	15	Six tungsten hairpin filaments suspended in grid form in one plane. Height of filaments 120 mm. Volts 105. Matches the colour of the 10 candle pentane lamp.
2	4·8 (Carbon basis)	11	1,700°C.	15	Single carbon loop filament in one plane. Height of filament 110 mm. Volts 100.
3	3·9 (Carbon basis)	10	1,750°C.	32	Five tungsten hairpin filaments suspended in grid form in one plane. Height of filaments 120 mm. Volts 109.
4	3·3 (Carbon basis)	10	1,790°C.	15	Gem coiled metallised carbon filaments (usual pattern). Volts 100.
5	1·9 (Tungsten basis)	13	1,870°C.	12	Four tungsten hairpin filaments suspended in grid form in one plane. Height of filaments 70 mm. Volts 95.
6	1·5 (Tungsten basis)	10	1,950°C.	18	Four tungsten hairpin filaments suspended in grid form in one plane. Height of filaments 70 mm. Volts 105.

Absolute Value of the Unit of Candle-power.

The foregoing description deals with the method used to determine the ratio of the average value of one set of sub-standards in terms of another, and, therefore, deals only with relative values. The *absolute* values of candle-power of sets 1 and 2 have been assigned to them respectively by two independent comparisons with the Harcourt 10 candle standard lamp. Set No. 2 (consisting of Fleming-Ediswan lamps) was compared against the pentane lamp in 1904-6. It will be seen from Table I. that the light from set 2 was somewhat whiter than that from the pentane lamp. At the time, however, that the determination was made their colour formed a suitable intermediate step between that of the pentane lamp and the usual light sources which had to be tested. After eight years' use of these lamps, and in view of some small unexplained differences between the N.P.L. unit and that determined in other quarters, it was considered desirable to realise the unit anew. Set No. 1 of the sub-standards was, therefore, prepared, whose light was identical in colour with that from the pentane lamp. These lamps were compared against the pentane standard in the usual way over a period of about two years (1912-14). The

results were analysed by the method of least squares in order to determine the true value of the sub-standards when the pentane lamp was burning under standard atmospheric conditions. It is satisfactory to find from the subsequent intercomparison of sets 1 and 2 that the unit of candle-power realised on these two different occasions separated by a period of eight years agreed with each other to within less than 1 part in 1,000 ($0{\cdot}_{07}$ per cent.). The value of the unit originally realised in set 2 is that which formed the subject of agreement between the laboratories of this country, France and the United States in 1908. The value obtained in the original determination has, therefore, now been fully substantiated by the new determination, and set 1 of the sub-standards becomes a fundamental set whose constancy is well assured, and the determination of whose candle-power values is independent of any colour difference. Incidentally, it has been possible to redetermine the constants of the pentane lamp affecting its changes with varying atmospheric conditions. These results are discussed in detail later.

It must be admitted that it is to some extent a matter of good fortune that the two independent determinations of the unit have agreed to within such close limits, since the first determination was made by one observer only, and since there existed a small colour difference his readings might well have differed from the average by a few parts in 1,000. It so happens, however, that his difference from the mean was in this instance negligible, and the magnitude of the unit as determined 10 years ago remains unchanged.

The following points of general interest arising from these and allied measurements will now be discussed.

1. The order of agreement between different observers in the final candle-power value obtained for the high efficiency lamps (set 6).

(a) By the cascade method ;

(b) By comparison in one step.

2. The accuracy to which photometric readings of this description are repeated by any observer and the probable errors of the results.

3. The order of agreement with other National Laboratories in the values assigned to high efficiency glow lamps which

differ in the hue of their radiation from the flame standards from which the fundamental units are derived.

4. The constants of the pentane lamp with changes of humidity, barometer and temperature.

5. The constancy of the unit of candle-power maintained through the medium of electric sub-standards.

1. *Values of Candle-power Obtained for the High Efficiency Tungsten Lamps.*

The precise numerical values obtained for the individual lamps are not of immediate interest. For the purpose of this Paper each set of lamps may be considered as one lamp. If there was evidence in the course of the work that any one lamp was undergoing a progressive change it was discarded, so that the figures given may be assumed in each case to be those for a set of lamps whose candle-powers have remained unchanged during the comparison.

Throughout the work the average of all six observers has been taken as giving the true value for any set of lamps. For the purpose, therefore, of showing the differences of individual observers this average value is taken as 100 for each of the six sets of lamps measured by the cascade method. The observers are designated by the letters A, B, C, D, E, and F.

Table II. gives a summary of the results. Observer A compared set 2 against set 1 about eight times, and obtained a value of $99{\cdot}5_5$ for set 2 as compared with a mean value for all observers, of 100. He then compared 3 against 2, assuming *his own* value of $99{\cdot}5_5$ for set 2. In the same way 4 is compared against 3 and so on to set 6. The bottom line, therefore, gives the values which each observer obtains for set 6 in terms of set 1 as a result of his own independent descent through the intermediate steps of the cascade.

TABLE II.—*Summary of Cascade Observations, each Set being compared by each Observer against the one immediately above it.*

Observer.	A.	B.	C.	D.	E.	F.	Average.
Set 1.....	100·00	100·00	100·00	100·00	100·00	100·00	100
„ 2.....	$99{\cdot}5_5$	$99{\cdot}4_5$	$99{\cdot}9_0$	$99{\cdot}8_7$	$100{\cdot}8_4$	$100{\cdot}3_9$	100
„ 3.....	$99{\cdot}4_3$	$99{\cdot}3_5$	$99{\cdot}9_5$	$99{\cdot}9_8$	$100{\cdot}8_5$	$100{\cdot}4_3$	100
„ 4.....	$98{\cdot}9_5$	$99{\cdot}0_4$	$99{\cdot}8_4$	$100{\cdot}3_1$	$101{\cdot}3_8$	$100{\cdot}5_0$	100
„ 5.....	$98{\cdot}7_3$	$98{\cdot}7_0$	$99{\cdot}7_9$	$100{\cdot}4_5$	$101{\cdot}4_8$	$100{\cdot}8_7$	100
„ 6.....	$98{\cdot}5_4$	$98{\cdot}6_1$	$99{\cdot}6_9$	$100{\cdot}7_1$	$101{\cdot}4_8$	$100{\cdot}9_7$	100

Table III. gives the value which each observer obtains for set 6 when compared in *one* step against set 1 with the Lummer

Brodhun photometer. It will be seen in the last column that this direct comparison of set 6 with set 1, entailing the full colour difference, yielded as the mean of all observers a result differing by a little less than 0·3 per cent. from the comparison by the cascade method (Table II.). By reference to Table V. it will be seen that this is the order of the probable error of either result.

TABLE III.—*Summary of Direct Comparison of Set 6 against Set 1 by each Observer.*

Observer.	A.	B.	C.	D.	E.	F.	Average.
Set 6.....	$99{\cdot}0_8$	$99{\cdot}5_5$	$98{\cdot}9_5$	$99{\cdot}3_7$	$100{\cdot}9_6$	$100{\cdot}4_1$	$99{\cdot}7_2$

Judged from the physiological standpoint, it cannot be said with certainty that the cascade method (Table II.) should give a result which is more absolutely rigorous than the direct method (Table III.). The case is however, somewhat analogous to the comparison of a low with a high resistance direct, and through the medium of a series of resistances with intermediate values. The authors place more reliance on the cascade method because of the less consistent manner in which observers by the direct method have been found to repeat their individual readings over considerable intervals of time. This lack of consistency has not been found where the small colour differences of the cascade method are concerned. All the measurements with the direct step were taken over a relatively short period of time whilst those included in the cascade method extended over several years. The experience of the authors where larger colour intervals are concerned does not afford an assurance that the "day to day" probable error of the direct step would have been so small, had the measurements in this case also extended over a longer period.

It will be seen that the largest difference of any observer from the mean in the first result by the cascade method was about 1·5 per cent. in the case of observer E. His mean difference per step was, therefore, about 0·3 per cent. By the direct method in one step the greatest difference was again by observer E, viz., $1{\cdot}2_4$ per cent.

An examination of the differences from the mean result revealed by each observer for each successive step in the cascade is facilitated by Table IV.

TABLE IV.—*Showing how much for each Step in the Cascade each Observer differed from the Mean of all for that Step.*

Comparisons.	Observer.					
	A.	B.	C.	D.	E.	F.
	%	%	%	%	%	%
Set 2 against Set 1	$-0\cdot4_5$	$-0\cdot5_5$	$-0\cdot1_0$	$-0\cdot1_9$	$+0\cdot8_4$	$+0\cdot3_9$
„ 3 „ „ 2	$-0\cdot1_2$	$-0\cdot0_7$	$-0\cdot0_9$	$+0\cdot0_9$	$+0\cdot0_4$	$+0\cdot0_6$
„ 4 „ „ 3	$-0\cdot4_8$	$-0\cdot3_4$	$-0\cdot0_6$	$+0\cdot3_5$	$+0\cdot4_8$	$+0\cdot0_5$
„ 5 „ „ 4	$-0\cdot2_2$	$-0\cdot3_4$	$-0\cdot0_5$	$+0\cdot1_4$	$+0\cdot1_0$	$+0\cdot3_7$
„ 6 „ „ 5	$-0\cdot1_9$	$-0\cdot0_5$	$-0\cdot1_0$	$+0\cdot2_6$	$+0\cdot0_2$	$+0\cdot1$
Mean	$-0\cdot$	$-0\cdot2_8$	$-0\cdot0_6$	$+0\cdot1_4$	$+0\cdot3_9$	$+0\cdot1_9$
Total (cascade)	$-1\cdot4_6$	$-1\cdot3_9$	$-0\cdot3_1$	$+0\cdot7_1$	$+1\cdot4_8$	$+0\cdot9_7$
Set 6 agnst Set 1(direct)	$-0\cdot6$	$-0\cdot1_7$	$-0\cdot7_7$	$-0\cdot3_5$	$+1\cdot2_4$	$+0\cdot6_9$

It will be seen from this table that there is, as would be expected, a general tendency for each person's observations to be either higher or lower than the average of all observers throughout the whole series. The amounts by which the determinations of any one observer are high or low for the different steps varied slightly, but no more than would be expected from Table V., showing the probable errors of the comparisons.

Observer E obtained rather a large difference from the mean, in the comparison of set 2 against set 1, but this appears to be the only slightly anomalous result in all the 30 cascade determinations.

The table shows clearly that even the small colour differences represented by a change of temperature of the filament of about 45°C. give rise to differences, though small, in the personal judgment of the various observers. Furthermore, when any one observer's differences for the five steps of the cascade are added together the total difference does not entirely accord with the results of that observer's *direct* comparison through the large colour step. (Compare the last two lines of the table.) For observers E and F the sum of the differences for the five small steps gave, it is true, nearly the same figure as for the one large step, but in all other cases the differences were appreciable, observers B and D especially showing unexpected results in the direct comparison. This illustrates what was stated earlier as to the more erratic nature of results obtained with large colour steps. The experience obtained with these and other comparisons is that whereas an observer may be relied upon for constancy of judgment in measuring with small colour differences, the same *constancy* of judgment is not obtainable with large ones.

Efforts were made to obtain a comparison between sets 1 and 6 by means of the flicker method. Several flicker photometers were tried by four of the observers, but the precision of the measurements was found to be so inferior to that obtainable with the Lummer Brodhun photometer that after much time had been spent the attempt was abandoned. The authors were unable to experience the sensitivity claimed for the flicker photometer by some other experimenters, and the results they obtained with it were of quite a different order of accuracy from those forming the subject of this Paper. Moreover, it is found that the fatigue occasioned by continuous photometric observation throughout a whole day necessitated by these standardisations was very serious with the flicker photometer, and tended still further to diminish the sensitivity.

2. *Probable Error of the Results.*

In computing the probable error in the value attributed to a lamp by a set of observations, the error may be regarded as arising from three distinct causes.

First, there is the error due mainly to the lack of sensitivity of the photometer, resulting in a succession of readings being obtained which differ slightly amongst themselves—all other variables being constant for the short period during which the lamp is being photometered. Five or more photometric settings have always been made by each observer on each occasion on which the lamp is tested. The usual probable error in the arithmetical mean of these five settings in any step of the cascade is 0·2 per cent. in candle-power, whilst for the direct step with the large colour difference it is 0·3 per cent. The ultimate probable error in the unit of candle-power held in a batch of lamps due to this cause is, however, very small, being reduced approximately in proportion to the square root of the total number of measurements made. This number being very great the error becomes entirely negligible.

The second source of error, called here the "day-to-day error," produces differences in the value obtained for the same lamp as measured from day to day. These discrepancies may be ascribed to the slight differences in the mechanical or electrical adjustments of the lamps, to small changes in the lamps themselves or to variations of the observers' judgment due to physiological or psychological causes. The probable error due to these causes is kept small, by each observer making his measurements on a sufficient number of different days. It

may amount to $0{\cdot}0_4$ per cent. for the determination of the values of one set of lamps in terms of an adjacent one in the cascade series. In computing this error it would not be right to regard the causes as entirely mechanical in character. If they were, the probable error would be much reduced by the fact that there were at least 10 lamps in each set. It is necessary to regard at least a portion of the error as due to physiological causes the effect of which is not reduced by having a number of lamps, but only by increasing the number of observers. In order to avoid too sanguine a figure for the "day-to-day" error, its value has been computed on the assumption that errors under this section are entirely physiological in origin.

The third and most important source of error lies in the consistent differences which exist between various observers, due to colour differences. In the cascade method of bridging the step from the lowest to the highest efficiency standards the errors are additive from one step to the next, and the final probable error has been calculated from the final differences tabulated in line 7 Table IV.

The amount of the probable errors ascribed to each cause for each step of the cascade is given in Table V. It will be seen that the probable error of the final result—viz., of the unit represented by set 6 in terms of set No. 1 is of the order of $0{\cdot}3_5$ per cent. both by the cascade and by the direct methods.

TABLE V.—*Giving the Probable Errors of the Various Steps of the Cascade and Direct Comparisons.*

Comparison.	Probable errors.			Total probable error in mean of set.
	Bench error.	Day-to-day error.	Error due to differences between observers.	
	%	%	%	%
Pentane lamp against set 1	0·01	0·05	0·00	0·05
Set 2 against Set 1	0·01	$0{\cdot}03_5$	0·15	$0{\cdot}15_5$
„ 2 „ „ 2	0·01	$0{\cdot}03_5$	$0{\cdot}04_5$	0·06
„ 4 „ „ 3	0·01	0·04	0·12	0·13
„ 5 „ „ 4	0·01	0·03	0·11	$0{\cdot}11_5$
„ 6 „ „ 5	0·01	0·02	0·04	$0{\cdot}04_5$
Total probable error by cascade method Set 6 against Set 1	0·02	0·07	0·36	0·37
Total probable error by direct method, Set 6 against Set 1	0·02	0·06	$0{\cdot}22_5$	0·23

3. *Intercomparison with other National Laboratories of the Unit of Candle-power in White Light.*

Having fixed the value of the unit in white light by the cascade method, it became of interest to know what order of agreement existed between it and similar units evolved at the National Laboratories of France, Germany and the United States. A set of 14 tungsten filament sub-standards with a specific consumption of about 1·5 watts per candle was prepared and sent to the Bureau of Standards, Washington. A similar set was prepared and sent to the Physikalisch-Technische Reichsanstalt, Berlin, and the Laboratoire Central d'Electricité, Paris. The lamps were tested both before leaving and after their return to England. At the same time a set of lamps was prepared at the Bureau of Standards and sent over here for test. Ordinary 100-volt metal filament glow lamps were used in these comparisons, and altogether about 18 months elapsed between the initial and the final comparisons. During this period some of the lamps appeared to change somewhat, and differences to the amount of about 0·5 per cent. were observed. In order to eliminate, as far as possible, the effect of these changes, it was assumed that when comparisons between any two laboratories were made over a short period of time no intermediate change in the lamp was likely to have taken place. When, however, the comparisons were spread over a long period of time the N.P.L. value assigned to any lamp was taken as the mean of the initial and final measurements on it.

The following table shows the value of the N.P.L. unit in terms of that of the Reichsanstalt, Bureau of Standards and Laboratoire Central, deduced from the mean of all the lamps tested :—

TABLE VI.

——	Ratio.	Difference from accepted value.
Bureau of Standards/N.P.L.	$1{\cdot}00_{55}$	$+0{\cdot}_{35}$ %
Reichsanstalt/N.P.L.	$0{\cdot}90_{02}$	$0{\cdot}_{0}$
Laboratoire/N.P.L.	$1{\cdot}00_{2}$	$+0{\cdot}_{2}$ %

A higher accuracy than 0·25 per cent. cannot be claimed for these intercomparisons, and the results, therefore, show that within the limits of accuracy of the experiment, the units of candle-power at the four laboratories realised in lamps operat-

ing at 1·5 watts per candle are in virtual agreement. The chief interest of the result lies in the fact that different methods were used at the several laboratories for carrying out the comparison. At the Laboratoire Central the comparison was made by measuring the lamps directly against standards of about 3·5 or 4·0 watts per candle—thus the large colour difference was bridged in one step. The Laboratoire Central values given are based on tests by an observer with normal colour vision. A second observer with abnormal colour vision obtained readings differing by 2 per cent. from these. By the method adopted at the Reichsanstalt a faintly coloured blue glass is placed between the photometer and the lamp of lower efficiency of the two under comparison so as to equalise the colours. This blue glass is calibrated once and for all, and the constants so determined are used in deducing the candle-power of the high efficiency lamps on test.

The values assessed by the Bureau of Standards are the means of comparisons taken (*a*) directly and (*b*) by means of a blue glass calibrated photometrically. In this calibration the mean of a number of observers was taken in fixing the transmission constants of the glass.

4. *The Redetermination of the Constants of the Pentane Lamp.*

The second realisation of the unit of candle-power from the pentane lamp, referred to on p. 263, has afforded an opportunity of re-examining the constants of the pentane lamp in regard to changes of humidity and barometric pressure. As in the previous determination,* the comparisons extended over two summers and winters, and by this means as wide a range as possible of humidity was secured. The conditions for the use of the lamp were similar to those existing previously, except that a larger number of observers took part in the measurements, and there was no colour difference between the light from the pentane lamp and that from the sub-standards compared against it. An Assmann ventilated hygrometer was used in the tests, and all humidities are expressed in terms of its indications. In view of the similarity of the experiments to those which have been already described, it is not necessary to give the actual observations in detail here. They have, however, been carefully analysed in the usual way by the method of least squares, and the constants for variation of

* C. C. Paterson, "Investigations on Light Standards," "Journ." I.E.E., Vol. XXXVIII., p. 274.

humidity and for barometric changes are compared below with those previously obtained.

If C.P.=the candle-power of the pentane lamp, e = the humidity in litres of water vapour per cubic metre of moist air, b=barometric pressure in millimetres, then

(1904-06 determination) $C.P. = 10\{1+0{\cdot}0066(8-e)-0{\cdot}0008(760-b)\}$ . . (1)

(1912-14 determination) $C.P. = 10\{1+0{\cdot}0063(8-e)-0{\cdot}0008_5(760-b)\}$ . . (2)

These two results may be regarded as practically identical, and are supported by those obtained by Messrs. Butterfield, Haldane and Trotter.* with whose formula the second one given above is identical.

A number of different determinations by various observers have now been made of the constants of the pentane lamp for changes of humidity and atmospheric pressure, and these are scheduled in the following table :—

TABLE VII.

Observer.	Date of publication.	Lamp.	Value of constants.	
			Humidity.	Barometric pressure.
a Liebenthal	1895	1 c. pentane	0·0057 †	0·0005
b Paterson	1904	10 c. ,,	0·0066	0·0008
c Dow	1906	,, ,,	0·0071	$0{\cdot}0008_5$
d Rosa and Crittenden	1910	,, ,,	$0{\cdot}0056_7$	0·0006
e Butterfield, Haldane and Trotter	1911	,, ,,	$0{\cdot}0062_5$	0·0008
Paterson and Dudding	1914	,, ,,	0·0063	$0{\cdot}0008_5$

a Liebenthal, "Zeitsch. für Instrumentkunde," 1905, p. 157.
b C.C. Paterson, "Proc." I.E.E., 1907, p. 271.
c J. S. Dow, "Elec. Review," Sept. 28, 1906.
d Rosa & Crittenden, "Trans." Ill. Eng. Soc., N.Y., 1910, p. 753.
e Butterfield, Haldane and Trotter, "Journ. Gas Lighting, 115, p. 290, 1911.

The tentative figure of 0·006 given by Rosa and Crittenden for the barometric correction must not, as stated by them in their Paper, be regarded as accurate, and the figures obtained by Liebenthal were for a 1 candle lamp of entirely different construction. Hence, neglecting these, it will be seen that

* Butterfield, Haldane and Trotter, "Journ. Gas Lighting," 115, p. 290, 1911.

† Based on a normal humidity of 8 litres per cubic metre.

there is general agreement as to the correcting factor to be applied for barometric changes. There is not, however, the same close agreement between the determinations of the humidity factor. Although the actual candle-power differences represented by the discrepancies between these different constants are small over the ranges of humidity met with in this country, the accuracy of most of the determinations would have justified the expectation of closer agreement in the value of this factor.

The work of Rosa and Crittenden at the Bureau of Standards, Washington, is of the highest accuracy, and the difference between their value and those determined in this country by the authors and by Messrs. Butterfield, Haldane and Trotter cannot in either case be ascribed to experimental error. In the publication of their work* Rosa and Crittenden make two suggestions as to how the difference might be accounted for, but they had not then before them the confirmatory evidence of the more recent investigations in progress in this country at the time they wrote.

It would seem that a definite difference in the conditions in Washington and London must be looked for to account for the discrepancy, and the present authors think it will be found to lie in the fact that the pentane lamp really has a temperature coefficient, but that its effect is almost entirely masked by the method generally used of determining and applying the humidity correction. In a more recent publication of the Bureau of Standards,† Messrs. Crittenden and Taylor suggest the possibility of a temperature coefficient, but they do not discuss its bearing on the constant for the humidity correction. In their earlier publication also they mention the effect which the temperature may have on the humidity correction, as determined in the earlier N.P.L. investigation.‡

The authors have not been able to determine the temperature coefficient of the pentane lamp. The difficulties found by all investigators, and mentioned by Crittenden and Taylor, of changing humidity and temperature independently have pre-

* *Loc. cit.*

† "The Pentane Lamp as a Working Standard," Crittenden and Taylor, "Bulletin" of the B. of S., Vol. X., p. 410.

‡ In a still more recent publication ("Bull. B. S.," Vol. X., p. 574), received after this Paper was written, the same authors refer to the subject again, and make the suggestion which has been elaborated here, but without giving actual figures connecting humidity and temperature at Washington. If these are obtainable and could be compared with those given here, the validity or otherwise of the suggested explanation would be established.

vented this. Nor is the matter of sufficient importance to justify the prolonged and difficult investigation which would be necessary. On the assumption, however, that there is a temperature coefficient the work already done enables an estimate to be made of its influence.

If it may be assumed that the candle-power of the pentane lamp under various atmospheric conditions is given by

$$\text{Candle-power} = 10 + A(8-e) + B(760-b) + C(*12-t), \quad . \quad . \quad (3)$$

where A, B and C are constants.

e=litres of water vapour per cubic metre,
b=height of barometer in millimetres.
t=temperature in °C.

The method of least squares enables us to determine the most probable value of the constants A, B and C from a large number of simultaneous observations of the variables, candle-power, humidity, barometer and temperature provided that :—

(*a*) All values of humidity, barometer and temperature are equally likely to occur conjointly ; or (*b*) if a definite relation exists between any of the variables not satisfying condition (*a*).

An examination of the observed values of humidity, temperature and barometer, reveal that condition (*a*) is fulfilled by the values of humidity and barometer or by the values of temperature and barometer, but that the humidity and temperature values satisfy neither condition (*a*) nor condition (*b*). This last fact is illustrated in Fig. 4, where the simultaneously observed values of humidity and temperature obtained whilst observing the candle-power of the pentane lamp both in 1904-06 and 1912-14 are plotted. It is seen that a *loose* general relation exists between these variables, the lines A and B representing the best linear relation for the two sets of experimental data, and so to some extent condition (*b*) is fulfilled.

If, now, it is assumed that the linear relationship shown by lines A and B in Fig. 4 actually exists between these values of humidity and temperature, the following are the values which result for the coefficients A, B and C in the above equation (3) :

$$\text{C.P.} = 10 + 0{\cdot}187(8-e) - 0{\cdot}008_5(760-b) - 0{\cdot}076_5(12-t). \quad . \quad (4)$$

Two facts should now be observed. Firstly, the temperature and humidity effects act against one another, and in practice it is the difference between the two which is operative. Secondly, as at Teddington, an increase of one unit of water

* The normal temperature of 12 deg. is chosen because it corresponds with the previously fixed normal humidity of 8 litres per cubic metre.

vapour is accompanied on the average by a rise of $1{\cdot}6_2$°C., the combined humidity-temperature coefficient becomes

$$0{\cdot}187—(0{\cdot}0076_5\times 1{\cdot}6_2)=0{\cdot}063,$$

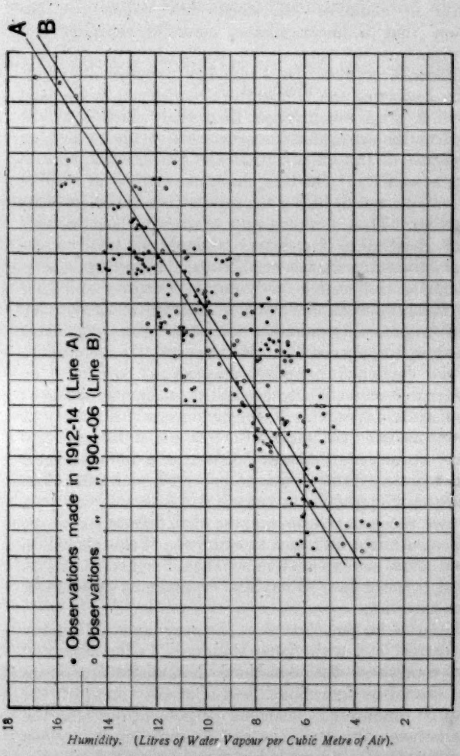

A
B
• Observations made in 1912-14 (Line A)
○ Observations " " 1904-06 (Line B)
Humidity. (Litres of Water Vapour per Cubic Metre of Air).
Temperature (C°).

FIG. 4.—DIAGRAM SHOWING VALUES OF THE TEMPERATURE AND HUMIDITY OF THE AIR OVER THE PERIOD DURING WHICH OBSERVATIONS WERE MADE ON THE CANDLE-POWER OF THE PENTANE LAMP.

viz., the coefficient in equation (2). Now the coefficient given by Rosa and Crittenden is $0{\cdot}056_7$, and it is readily seen that this would result from a prevailing climatic condition in which

an increase of one litre of water vapour per cub. metre corresponds with an increase of 1·7°C., instead of the 1·6$_2$°C. observed at Teddington. It is obvious that too great a significance must not be attached to the actual values of the coefficients in equation (4), since they depend on the assumption that a linear relation connects humidity and temperature.

The difference between the "humidity" coefficients determined (by neglecting the temperature coefficient) in England and America being much larger than that which could be attributed to the error of the experiments, tends to support the suggestion of the authors that the pentane lamp has a temperature coefficient, but that the usual method of making the observations and deducing the results does not allow of its determination. Thus, it would appear that the "humidity" coefficients determined for flame standards are really combined humidity-temperature coefficients. Whenever, therefore, a lamp is used under conditions of humidity and temperature which approximate to those existing at the locality where the original determination was made, the constant so determined will apply rigidly. If, however, the determination of the combined humidity-temperature coefficient be made under different climatic conditions, a slightly different constant may be expected. If, for instance, the humidity at Washington tends on the whole to increase at a different rate with temperature than it does at Teddington a different factor for the combined effects would be expected to result. The authors are not in a position to know if this is actually the case, but it is not unreasonable to suppose that differences of the order indicated might be found to exist, and if this should be so it would afford an explanation of the difference which has been found between the "humidity" coefficients determined in the two localities.

The conclusion is, that if work of the very highest accuracy is to be carried out with flame standards under abnormal humidity conditions the combined humidity-temperature coefficient should be determined for the locality in which the work is to be conducted. It should be pointed out that the difference between the American and English determinations for the pentane lamp amounts to less than 1 per cent. in candle-power for a rise of humidity of 10 litres per cubic metre above the normal. Values have seldom been observed greater than this in Teddington. The table at the conclusion of the Paper

by Crittenden and Taylor shows the average humidities in Boston and New Orleans to be 9·9 and 19·1 litres per cubic metre respectively. Where such large differences exist in climatic conditions the question of the variation of the combined temperature-humidity coefficient might with advantage be further investigated.

(6) *The Constancy of the Unit of Candle-power held in Electric Sub-standards.*

The continual use which has been made of electric sub-standards since the first determination of the unit of candle-power in 1904, has afforded opportunities of watching the behaviour of such lamps and particularly of observing their constancy.

Several observers have written on this subject, amongst whom are J. A. Fleming, C. H. Sharp, P. S. Millar, E. B. Rosa, G. W. Middlekauff,* and others, and all witness to the constancy of properly prepared and seasoned electric sub-standards. Their observations have been mainly concerned with carbon filament glow lamps, and the present authors are able to endorse the views they express. In order to keep records of their behaviour, an annual analysis is made of all the photometric records where sub-standards have been used during the year. When an ordinary routine standardisation has to be made, three to six sub-standards are put on the bench in turn, and thus for each day's work a value is obtained for any one standard lamp in terms of others of the same set. The analysis of such results over a year's working indicates if any individual lamp shows signs of differing from the mean of the others. It has been unusual to find that a lamp has appeared to change during 12 months by more than 0·1 per cent., and the majority of them show no change which can be detected. If any lamp in a set shows a difference from the mean of 0·1 per cent. no change in the value assigned to it is made until such difference is repeated in the following year's analysis. Of the one or two lamps whose values have had to be adjusted in this way, some appear to have risen and some to have fallen in candle-power, and there is no sign that any fundamental set of lamps is undergoing a progressive change. This can be said equally of the tungsten filament lamps as of those with carbon filaments.

* Fleming "Proc." Brit. Assn., 1904. Sharp and Millar, "Trans." Ill. Eng. Soc., N.Y., June, 1910. Sharp "American Gas Inst.," Oct., 1913. Rosa and Middlekauff, "Proc." Amer. Inst. Elect. Eng., July, 1910, p. 1911.

A set of 1·5 watts per candle tungsten standards (set 6) has to be used nearly every day of the week, and a fundamental as well as a working set of these lamps is, therefore, kept. The working set fell in candle-power about 0·5 per cent. during the continuous use of the past two years. As the fundamental set will only be used perhaps two or three times a year, its constancy for very many years is assured. It could at any time be compared against set 5 (2 watts per candle), which has never to be used in ordinary routine work, and which, as a matter of fact, has not been used since the values of its individual lamps were first fixed by the cascade method. Similarly, the use of set 1 (matching the pentane lamp in colour) will never be necessary in ordinary work because of the redness of the light. When it is remembered that the filaments of these lamps are of tungsten operating at the low efficiency of about 7 watts per candle, and that they only require to be used for exceptional reference purposes at intervals of several years, there would seem to be every reason for expecting the unit of candle-power to be maintained constant by means of them for an indefinite period.

The authors desire to place on record their obligations to Dr. R. T. Glazebrook, C.B., F.R.S., Director of the National Physical Laboratory, and on behalf of the Laboratory to acknowledge the generosity and help of the General Electric Co. and the Osram Lamp Works for the many expensive and special standard lamps which they have made and presented in connection with this work.

DISCUSSION.

Mr. A. P. Trotter said that an incidental result of the work described in this Paper was to reveal the precision with which photometric measurements are now made, and to raise the status of that operation. He wished to confine his observations to the moisture coefficient for the pentane standard, and to the possibility of the existence of a temperature coefficient. *A priori* considerations seemed to suggest that an appreciable temperature coefficient would exist. Pentane is a highly volatile substance. In the absence of a wick, the feed depends on the volatilisation of the pentane, and on the siphoning of the heavy vapour. Small departures from the standard pattern materially altered the flow of air and vapour by introducing thermal differences. But though a temperature coefficient has been looked for it has not been found experimentally. The authors suggest that the lamp really has a temperature coefficient, but that "its effect is almost entirely masked by the method generally used for determining and applying the humidity correction," and that "temperature and humidity effects act against one another." They say that if "the humidity at Washington tends on the whole to increase at a different rate with temperature than it does at Teddington, a different factor for the combined effects would be expected to result." This looks at first as though it were suggested that European laws of

hygrometry and dew-point did not hold good in America. The explanation probably is to be found in the degree of saturation. At low temperatures the moisture in the air is low in both climates, but at high temperatures the air is less saturated in Washington than at Teddington. In that case the effects of temperature would be more marked at the former place than at the latter. Both the variations of atmospheric pressure and of moisture in the case of the American observations and of those at the National Physical Laboratory were the result of natural metereological changes. The range of humidity at Washington varied from 0·4 per cent. to 2·7 per cent. by volume. The variations in the case of the experiments of Butterfield, Haldane and Trotter were for the most part produced artificially, and the ranges were much larger. Atmospheric pressure, humidity and vitiation by CO_2 were varied separately, one effect at a time, the other two remaining normal. Atmospheric pressure was varied from 450 mm. to 1,000 mm. Moisture varied, in the case of the pentane lamp from 0·73 per cent. at a temperature of 6·6°C. to 4 per cent. at a temperature of 33·2°C. It was not considered safe to use the pentane lamp at a higher temperature, but other experiments were carried on at 44°C. The moisture was produced by a flow of steam from a boiler outside the room, and the temperature was raised by electric heaters and gas burners. A saturation of about 90 per cent. was reached. After applying the corrections for atmospheric pressure and for CO_2, both of which had been well ascertained, it was found that the difference of light due apparently to moisture could be accounted for by a correction employing the coefficient 0·00625. The mean residual error was about 1·3 per cent. Experiments were made at temperatures varying from 8°C. to 20°C., the other conditions remaining approximately normal, the humidity varied only from 0·84 to 1·21 per cent., but without producing any material alteration in the light. Since the humidity at nearly all temperatures may be assumed to have been greater in the British than in the American investigations, it follows that the temperature will have had less relative effect on the British than on the American humidity coefficient. Hence, if the authors' assumption that there should be a temperature correction is valid, it is obvious that the British humidity coefficient is more nearly correct for humidity *per se* than the American. The majority of the observations from which the humidity co-efficient has been deduced here, and probably in America also, were made at higher temperatures than 15°C. (which presumably would be the accepted standard temperature), consequently if the temperature effect is in the opposite sense to the humidity effect the true humidity coefficient would be something higher than the British figure of 0·00625. The difference between this figure and the American coefficient 0·00567, viz., 0·00058, may be taken as a first approximation for the temperature coefficient. It cannot, however, be expressed in terms of temperature without more information about the temperature at which the American observations were taken. Experiments are needed to settle this. They should be conducted in winter in a chamber provided with the necessary arrangements for withdrawing the products of combustion, and with means for raising the temperature while maintaining as low a degree of humidity as possible.

Mr. J. S. Dow congratulated the authors on the painstaking work described in the Paper. The fundamental difficulties involved in comparing heterochromatic sources of light have not yet been overcome, nor the complex problems settled; but such devices as the use of coloured screens seemed to assist those who had not much experience of such work, and no doubt a series of carefully prepared standards, such as those mentioned by the authors, would be of service. Nevertheless, it is doubtful if such methods are worth the trouble involved in the case of fairly expert operators. The various observers using the cascade method did not seem to agree much better than those relying on direct comparison. If due regard is paid to the elimination of personal errors by taking sufficient observations, direct com-

parison is the simplest and perhaps the most reliable method. The cascade and other methods are of scientific interest as a "check." In view of the minute differences with which the authors worked he would like to ask if an appreciable difference in heterochromatic work could not be obtained by pushing in and out the telescope of the Lummer-Brodhun photometer. Some experiments he made about eight or 10 years ago had led him to think that an appreciable difference could be obtained in this way. Also, in comparing, say, a tungsten lamp with the pentane, a distinct difference might be noted in the results obtained with three good photometers of different types, taken at random; for the size of the photometric field affected the physiological impression of brightness. The authors had referred to the effect of humidity on the pentane standard. Some very good work on this subject had been done in the States where the wide climatic variations were favourable to such researches. These results seemed to confirm Mr. Paterson's investigations; the chief difficulty in America at that time seemed to be to get exactly reproducible lamps. He would like to know how close the lamps in this country now agreed. Some very careful comparisons between the flicker and the equality of brightness photometers had recently been made in the United States, and the tendency is to credit the flicker photometer with greater reliability in colour work. Personally he would prefer an instrument of the ordinary type—except possibly when very violent colour-contrasts were in question.

Mr. A. Campbell also thought that the figures in the Paper did not justify the conclusion that the cascade method was much superior to the direct comparison of the sources.

Mr. T. Smith (communicated remarks) asks the authors if, apart from the reduction in the differences between different observers, any advantage is obtained by the adoption of the cascade method regarding it solely as a means of finding the candle-power of lamp 6 in terms of that of lamp 1. It would be of interest to see a curve plotted showing the relation between the candle-power found for lamp 6 from that of lamp 1, and the number of intermediate lamps used in making the comparison. The figures given in the Paper would determine the extremities of such a curve, but further information was required before more than a conventional meaning could be assigned to the figures obtained for the candle-power of the lamps higher in the series. The curve suggested should help to show whether direct comparison or cascade measurements were to be preferred in the determination of standards. A comparatively small number of observations would determine the form of this curve. The photometric comparison of lights of different colours may be affected by the chromatic aberration of the eye. The difficulty of deciding when two colours are equally bright may be partly due to the eye requiring to accommodate itself so as to bring each colour in turn into sharp focus on the retina. It would be of interest to compare the results described in the Paper with others made with the same lamps and by the same method, but with a lens placed in the photometer head just in front of the eye and so constructed as to cause the photometer contrast pattern to be in good focus on the retina simultaneously for practically the whole extent of the visible spectrum; no magnification would be produced. The lens would be correct for all observers, and the results obtained would be of interest, apart from the possibility of reducing the uncertainty in measurements of candle-power. Perhaps some Fellow of the Society could supply quantitative details of the chromatic aberrations of the eye from which such a lens could be computed. Such details did not appear in the standard works on Physiological Optics.

Mr. J. Guild (communicated): As pointed out by the authors, it appears to be necessary even in comparing ordinary "white" lights to stipulate normal colour vision on the part of the observer, and the cascade method of subdividing colour differences, while minimising the probable errors of observation, leads apparently to the same result with a particular observer as should be obtained by direct comparison. The authors quote results from

six observers, but to what extent one is justified in assuming the average colour vision of that group to agree accurately with that of another similar group which might be chosen at some other standardising institution it is difficult to say, and it seems highly desirable to have some method by which light may be measured absolutely rather than by comparison with arbitrary standards. In the "Proceedings" of the Royal Society (Vol. LXXXVII., 1911, p. 275) Dr. Houston proposes to reproduce the selective sensitivity of the eye to energy of different wave-lengths by a suitable light filter of which the transmission curve is similar to the sensitivity curve of the normal eye and to measure the transmitted energy thermo-electrically. The arrangement virtually constitutes a standard "eye" which has the property, not possessed by natural eyes, of measuring the intensity of the light which it receives. Have the authors considered the proposal? The authors state the approximate illumination at the photometer head throughout their work. This is a useful practice. As is well known, the sensitivity curve of the eye for the visible spectrum varies with the intensity of the incident light, the red end being penalised as the general brightness is reduced. Hence, in comparing two sources of different hue, a lower value will be ascribed to the redder light the further we remove the sources from the photometer. The cascade method of subdividing the colour gap will not get over this, and it would be of interest to know what range of illumination at the photometer might be employed in the subsequent use of the authors' sub-standards without introducing errors comparable with the degree of accuracy aimed at. Pulling out the draw tube of the photometer, by reducing the brightness of the retinal image, would produce the same result as moving the sources further off, and this might account for the effect observed by Mr. Dow, since the angular size of the field of the photometer seems to preclude the explanation that this is due to the varying colour sensitivity of different parts of the retina. Errors of this description may be wholly insignificant within the range of feasible experimental conditions, but the advent of such high precision in photometry makes an investigation of the possible magnitude of such errors desirable. Until such an investigation is made it appears to be a necessary precaution, when using any sub-standard source of light, to use it at a constant distance from the photometer and with the same type and size of photometer as when the lamp was standardised.

Mr. C. C. Paterson, in replying for the authors, noted that Mr. Trotter agreed that it was possible that the relative degree of saturation in America and London would account for different correcting coefficients, if it was found that the pentane lamp actually had a temperature coefficient. Mr. Trotter seemed to imply that from his experimental results covering a larger range of humidity and temperature it might be possible to deduce the temperature coefficient. In order that this could be done one wanted as many observations where there were low humidities and high temperatures as there were observations in which both humidity and temperature were high. This is a condition which up to now it has been found impossible to obtain, and it would seem necessary to fall back on the method suggested in the Paper in which the conditioned relationship between humidity and temperature was determined, and by using this in the method of least squares to obtain an approximation to the temperature coefficient. The authors do not think it possible to determine the magnitude of the temperature coefficient in the way suggested at the end of Mr. Trotter's remarks. As the combined humidity-temperature coefficient is probably the result of the algebraic sum of a positive temperature coefficient and a negative humidity coefficient, the actual temperature and humidity-coefficients would be expected to be of quite a different order from the combined one. As shown in the Paper, the humidity-coefficient of +0·187, combined with a temperature coefficient of −0·076, would fit the experimental results quite well instead of the present combined coefficient of +0·062. In reply to Mr. Dow and also to Mr. Campbell, the somewhat laborious cascade method of com-

parison was only suggested as a means for a central laboratory to establish and standardise its sub-standards of light of different hues, so that standard lamps could be issued to other testing-rooms the colour of whose light was suitable for the work done. It was quite true that the sets of high efficiency sub-standards now used at the National Physical Laboratory for such standardisations would have been given substantially the same candle-power values when standardised directly against the primary standards as they have actually been given by the cascade method. But this could not be foreseen. Nor would the former method have given sets of standards of intermediate colours, against which the high efficiency standards can be checked back at any time, or against which other lamps of intermediate efficiencies could be standardised. A set of high efficiency sub-standards whose values are known in terms of the average eye is essential, otherwise all ordinary standardisations and photometric measurements would have to be carried out with a large colour difference, which would involve serious errors unless all measurements were made by a large number of observers. Perhaps Mr. Dow had not kept sight of this in his observations. The question of the size of the photometric field raised by Mr. Dow is an interesting one. The authors had also made observations of this effect and found differences where the colour contrasts were very great and the distance of the eye from the photometer varied over wide limits. As Mr. Guild has pointed out, however, in an ordinary photometer the field of vision is relatively large. The result of this when working with the Lummer-Brodhun head is that an observer would not make a judgment of the field as a whole but would look at any one moment at one particular portion of it, and they would not have expected any different result if the field had been twice the size. As a matter of fact, in all measurements each observer adjusted the telescope so as to bring the field of view into proper focus, and differences due to variations in the size of field were not noticed. Where the colour differences are small, as in the cascade method, different photometers intelligently used on the substitution principle would probably agree. The authors are not able to say anything definite as to the reproducibility of the pentane lamp. All their work has been done with the standard at the National Physical Laboratory. Perhaps a rough figure of 1 per cent. might be taken for the closeness of reproducibility of the pentane lamp as at present constructed. In reply to Mr. T. Smith, it would be interesting to have the results he suggested; but as any one set of photometric standard glow lamps cannot be expected to keep constant within 0·1 per cent. if too large a number of observations are made with it, it is impracticable to extend the number of observations indefinitely. A very large number of observations would be necessary for the accuracy to be comparable with that given in the Paper. The "artificial eye," mentioned by Mr. Guild, is a very attractive subject. The authors have not aimed at such a device. Sensitiveness is one of the difficulties. Such an "eye" must be adjusted and calibrated in terms of a large number of observers before its readings can be taken as having a photometric value, and it is therefore not more fundamental than the procedure suggested by the authors. The phenomenon discussed in Mr. Guild's concluding remarks appears to be a Purkinje effect. Mr. Dow has shown that this begins to come in at illuminations of the order of 0·1 ft. candle. It is very seldom that illuminations at photometers approach such low values. Photometric work tends to be carried out at increasing rather than decreasing illuminations, about 1 to 2 ft. candles being usual.

XXI. *The Relative Losses in Dielectrics in Equivalent Electric Fields, Steady and Alternating.* *By* G. L. Addenbrooke, *M.I.E.E.*

Received February 3, 1915.

In "The Electrician" for March 1, 1912, I described, I believe for the first time, the general relations between the losses in dielectrics in fields of equivalent voltage, continuous and alternating (R.M.S.), the latter having a range of periodicities of from 1 in 4 seconds to 42 $\sim$ per second, the dielectrics being in all cases in intimate contact with the electrodes.

As $R=\frac{E^2}{W}$, the continuous-current resistance or the corresponding specific loss can be compared with the alternating actual or specific loss, under similar conditions as regards voltage, it being understood that with continuous P.D.'s the field is maintained long enough for the deflection to become steady or nearly so.

In a communication in May, 1912, to the Physical Society, it was shown that similar relations exist in the losses occurring across the films of moisture which are present on the surfaces of dielectrics.

In both these communications attention was drawn to the variations in the losses caused by slight differences in the amount of moisture present, both in the magnitude of the losses and in the forms of the diagrams obtained by plotting them.

Lastly, in "The Electrician" for January, 1913, I described further experiments showing in detail the actions occurring in a fairly high-class dielectric—viz., gutta-percha at very low periodicities up to 42 per second.

Part of the data given in this latter communication have since been used by Dr. Ashton in his Paper last session on dielectric theory. Working on lines developed by Maxwell, Hopkinson, Pellat and Schwidler, and extended by himself, Ashton appears to be able to reproduce my curves fairly closely for periodicities above about two per second. Below this frequency, however, the discrepancies increase. I have re-examined these results since, and have also submitted certain others to Dr. Ashton, and it does not appear that there can be any error in the observations which is at all likely to account for the differences.

Again, for frequencies *above* 6 *or* 8 *per second* the curves of

losses at different frequencies are very well represented by assuming a fixed loss added to a loss which varies directly with the frequency, or is constant per cycle whatever the frequency. This very simple formula seems to represent the losses from the above to very high frequencies with considerable accuracy, but below these frequencies it fails, and the fixed loss which must be assumed on this basis is always larger, and usually much larger, than that given by the equivalent steady current; in fact, the two seem with good dielectrics not to bear any relation to each other.

Nevertheless, in the course of measuring the alternating and continuous losses in a large number of dielectrics I almost subconsciously came to feel about two years ago that there was a direct connection, and I determined to look into the matter more closely.

It was obviously desirable to include results for a number of different dielectrics of widely different composition, and also to get the data on some simple and common basis, which would leave the ratios of the actions in the different dielectrics to stand out as clearly as possible. Finally, it seemed best to take the specific loss corresponding to a specific resistance of 1×10^{15} ohms per centimetre cube as a basis, and express all the losses as multiples of this. It seems better at this stage also to deal with both the continuous and alternating losses as dissipation of energy without postulating anything further as to whether it is of a conductive character or not.

As a commencement the continuous pressure loss is compared with the alternating loss at 40 $\sim$ only; the question of what the results would be had another periodicity been chosen will be considered later.

All the measurements, unless otherwise stated, were made by enclosing each dielectric in the form of a sheet, in a container, between mercury faces, at an average pressure of not less than 4 in. of mercury. The observations, where not otherwise specified, were all made at room temperatures, the two sets of observations on each substance one immediately after the other. The voltages used were between 100 and 120 to avoid any chance of heating or straining the dielectric, a point about which it is necessary to be most careful, particularly when the dielectrics are not of the highest class. The observations were made as described in the above communications with the latest form of my electrostatic wattmeter and a special form of reflecting electrostatic voltmeter.

In the tables the different dielectrics are arranged in the order of the specific resistances, at present their best known attribute, the values of which are given in the second column.

The third column gives the corresponding specific loss which corresponds with the conductivity.

The fourth column gives the corresponding specific alternating loss at 40 ∿ in the same units.

The fifth column gives the ratio of the specific continuous losses to the alternating for each separate substance.

TABLE I.—*Connection between Continuous and Alternating Losses* (40 ∿), *specific, in Dielectrics, at the Same Equivalent Voltages* (*C. and R.M.S.*, 100-120 *Volts*).

Solids. Substance and state.	C.C. resistance in ohms per cm.³.	C.C. conductivity 10^{-15} mohs as unity.	A.C. conductivity 10^{-15} mohs as unity.	Ratio of losses for each substance.
Blotting paper, room dryness.....	$1{\cdot}3\times10^{9}$	770,000	1,150,000	1 : 1·5
Celluloid, room dryness, 0·62 mm.	$1{\cdot}7\times10^{10}$	59,000	118,000	1 : 2·0
Blotting paper, dried in sun	$3{\cdot}2\times10^{10}$	31,000	71,000	1 : 2·3
(*a*) Celluloid room, dry 0·4 mm....	$3{\cdot}2\times10^{10}$	31,000	65,000	1 : 2·1
(*b*) „ „ 0·23 mm....	$5{\cdot}7\times10^{10}$	17,500	44,000	1 : 2·5
Rubber-covered wire	$1{\cdot}2\times10^{11}$	8,000	64,000	1 : 8·0
(*a*) Celluloid dried, 0·4 mm.....	$2{\cdot}9\times10^{12}$	340	6,400	1 : 19·0
(*b*) „ „ 0·23 mm.....	$4{\cdot}4\times10^{12}$	230	4,800	1 : 21
Rubber sheet water soaked, 12°C.	$2{\cdot}0\times10^{13}$	50	3,500	1 : 70
Crown glass, 11°C.	$3{\cdot}7\times10^{13}$	28	6,200	1 : 220
Rubber sheet, room dry, 12°C.....	$5{\cdot}0\times10^{13}$	20	1,800	1 : 90
Gutta-percha, 25°C.	$2{\cdot}0\times10^{14}$	5	1,750	1 : 350
Rubber sheets dried, 100°C. & 12°C.	$2{\cdot}6\times10^{14}$	4	1,600	1 : 400
Bohemian glass flask, 12°C.....	$8{\cdot}0\times10^{14}$	1·2	5,100	1 : 4,200
Gutta-percha, 11°C.	$1{\cdot}0\times10^{15}$	1·0	1,200	1 : 1,200
„ „ 8°C.	$1{\cdot}7\times10^{15}$	0·6	1,080	1 : 1,830
Hetero geneous Sol id.				
Micanite	$2{\cdot}25\times10^{14}$	4·4	7,500	1 : 1,700

TABLE II.—*Liquid Dielectrics.*

Substance.	C.C. specific resistivity ohms.	C.C. conductivity 10^{-15} mohs as unity.	A.C. conductivity 10^{-15} mohs as unity 40 ∿.	Ratio of losses C.C. to A.C. for each substance.
Petroleum, ordinary	$6{\cdot}0\times10^{10}$	16,600	33,000	1 : 2·0
Mineral oil and petroleum mixed	$2{\cdot}4\times10^{11}$	4,100	15,000	1 : 3·7
Olive oil, as obtained.....	$1{\cdot}0\times10^{12}$	1,000	3,200	1 : 3·2
Mineral oil, thin	$1{\cdot}5\times10^{12}$	650	3,250	1 : 5·0
Castor oil, medicinal	$2{\cdot}0\times10^{12}$	500	2,500	1 : 5·0
Mineral oil, thick (1).....	$2{\cdot}5\times10^{12}$	400	2,650	1 : 6·6
Above oil heated to 185°C. and cooled (2)	$6{\cdot}5\times10^{13}$	15	1,950	1 : 130·0

The chief points brought out by these tables are :—

(*a*) That the order of the alternating losses follows so closely the steady pressure losses that there can be little doubt that the two are different aspects of phenomena arising from the same cause, and this applies both to liquid as well as solid dielectrics.

Out of the 24 examples given in the tables there are only three exceptions—micanite, crown glass and thin blown bohemian glass. Micanite was specially chosen as a clearly heterogeneous substance. The mica in this has a very high specific resistance and very small alternating losses, but the shellac has a little moisture in it, and should show correspondingly high losses in an alternating field. If then the losses are due to the continuous movement of ions, or, possibly, of free electrons through the dielectric in a steady field, we should expect, in this case, that having for the most part to make their way deviously round the edges of the mica flakes, their motion would meet with an inordinate amount of resistance in comparison with the thickness of the dielectric. And this is just what we see happens. On the other hand, with an alternating field the motion being reciprocal and through relatively very short distances would be little affected by the presence of the mica, and would be relatively large. And this again is what we see takes place.

As illustrating this and some other effects, the following curves are interesting. The upper curve shows the relative losses, steady and alternating, in a sheet of celluloid of room dryness over a range of frequencies. The lower curve shows the effects when the same sheet of celluloid, which was 1·4 mm. mils thick, was sandwiched between two sheets of gutta-percha each 0·06 mm. thick. In the latter case it will be seen that loss due to the steady field is reduced practically to nothing, between this and 16 periods there is considerable reduction in the losses, but between 16 and 40 periods in each case the curves are practically straight lines and parallel to each other. The portion of the alternating losses, which is indicated by b, a constant and n the frequency or bn, is, therefore, sensibly the same in each case, and is unaffected by the presence of the thin sheets of gutta-percha. To get a correct result it is, of course, necessary to make a proper allowance for the presence of the gutta percha sheets which, increasing the thickness and introducing another substance of different dielectric constant, reduces the inductive effect by a small percentage. Below is

added a third curve to show the effect of drying. All these curves are to the same scale of ordinates. Nearly identical curves may be obtained by sandwiching a sheet of mica or gutta percha between two thin sheets of celluloid or blotting paper, the point being that if there is a serious obstruction

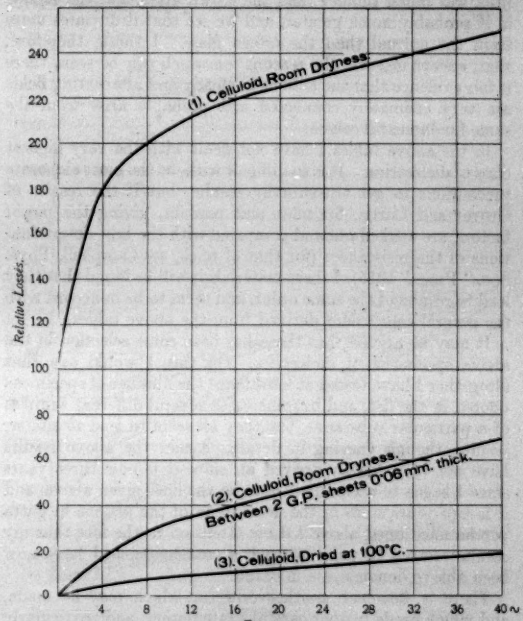


DIAGRAM I.

Celluloid (1·4 mils thick).

Comparative losses, alone and between thin G.P. sheets.

anywhere or of any kind, the normal relations between the losses in equivalent steady and alternating fields will be altered, and while the steady field loss will be reduced the alternating action will be less affected, so that the ratio of the two to each other is increased. I think, therefore, that we

shall not be wrong in attributing the differences observed with the two glasses in the above tables to a similar cause, when we consider that glass is naturally of a crystalline nature, and the great strains and deformation which must occur when it is blown or rolled in thin sheets. In these two cases the glass flask was much thinner than the crown glass, and the strains in it probably much greater, and we see that it deviates more from the normal than the crown glass. I think, therefore, that, except in cases the reasons for which can be seen, there is fair evidence that the actions in steady and alternating fields are very intimately connected and probably arise from the same fundamental causes.

In the above tables I have not dealt with the very highest class of dielectrics. It is so difficult without the most elaborate precautions to get trustworthy results, but if the results of Grover and Curtis, for mica and paraffin, giving the power factors, are worked out and compared with the best determinations of the resistances (for that of mica, *see* Campbell, Phys. Soc. "Proc.," 1913) of these materials it will be found that they lead to results of the same order, and seem to be in accord with the general conclusions derived from the above tables.

It may be argued that there has been some selection in the above specimens of dielectrics. On this I would say that altogether I have tested at least twice the number of specimens quoted in the list, and in some cases several different samples of a particular substance, but they all seem to lead to similar results, though varying in detail. Again, the above results have not been hastily arrived at, since it is now three years since I began to accumulate data on the lines given above, and it is two years since in the description of the actions in gutta percha mentioned above I drew attention to the fact that my results seemed to point to such a connection as I have now been able to demonstrate in detail.

There is, however, another criticism which may be made, and which needs further careful examination, and particularly as it gives an opportunity of bringing out some further and, as it seems to me, interesting and important points. It may be objected that while the results apply to the ratios of the losses, with steady pressures and at 40 $\sim$ the sequence might be by no means so obvious if some other periodicity were taken.

To deal with this point it is necessary to consider the losses not only at one but at several different frequencies. For this purpose I have selected five examples as typical. They in-

clude the first and last of the examples given above and three intermediate ones, including crown glass. The full curves for the losses in these are given below. As, however, owing to the greater range of the losses these cannot be got into a diagram to the same scale, and therefore a casual inspection of the curves does not convey a proper idea of the proportionality

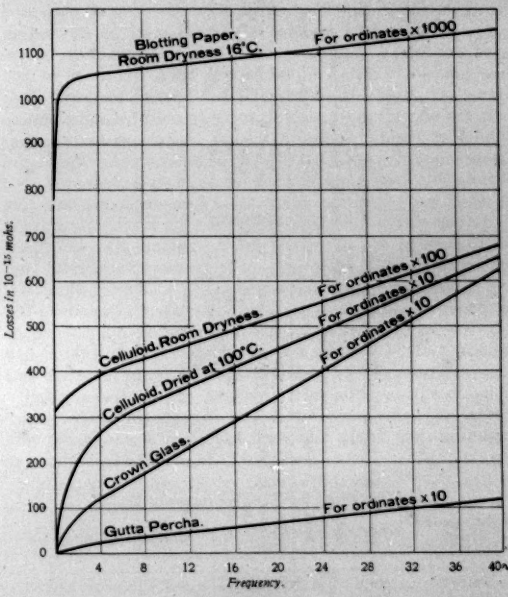


DIAGRAM II.

Losses in dielectrics in equivalent steady and alternating (R.M.S.) fields. Frequencies 1 in 4 secs. to 40 ∿.

of the actions. I subjoin below the data from which the curves were drawn.

The first point is that above 8 to 12 periods per second the rising curves are practically straight lines in all cases, and whether the dielectrics are good or poor. In this diagram the

TABLE III.—*Specific Losses in Dielectrics, taking the Loss through* 1×10^{-15} *Mohs as Unity.*

Substance and state.	Sect. Mils.	Temp. °C.	Losses, steady field.	Losses, alternating fields and frequency.								
				∼0·25	∼0·5	∼1	∼2.	∼4.	∼8.	∼16.	∼25.	∼40.
Blotting paper, room dry.	25	16	770,000	...	1,000,000	1,025,000	1,035,000	1,055,000	1,070,000	1,100,000	1,120,000	1,150,000
Celluloid, ditto	16	15	31,000	32,000	33,000	34,000	35,700	39,500	42,500	49,500	55,500	65,000
„ dried 100°C.	16	15	340	...	...	1,250	2,000	2,550	3,250	4,000	4,900	6,400
Crown glass	66	11	28	200	310	510	800	1,100	1,800	2,800	4,000	6,200
Gutta percha	2·5	11	1	10	19	36	75	165	330	550	800	1,200
					Liquids.							
Thin mineral oil	...	...	4,200	...	11,000	11,500	12,500	13,200	13,600	13,900	14,600	15,400
Castor oil	...	...	...	...	...	...	...	...	...	...	...	...
Mineral oil, dried	...	...	...	...	...	...	...	...	...	...	...	...

curves end at 40 periods, but for the past 18 months I have had an alternator capable of giving 100, 200 and 400 periods, and have made numerous tests with this. In all cases the curves continue sensibly as straight lines. Further, in one case, with celluloid, through the courtesy of the Research Department of the G.P.O., and with the help of their Mr. C. E. Hay, we have been able to carry the tests up to 2,000 $\sim$. As far as can be judged the curve is practically straight from, say, 12 $\sim$ per second up to this periodicity. In some cases I have also been able to fit results of my own on to the results of Fleming and Dyke (*see* their Paper I.E.E. "Proceedings," 1912), and as far as can be judged from such rough tests, since they do not give the ordinary resistances, my curves can be directly prolonged into theirs at frequencies of from 900 to 4,500. It seems clear, therefore, that this is a general law for dielectrics of all classes—viz., that above about 8 to 12 $\sim$ the curves of their losses at varying periodicity are sensibly straight lines at any rate up to telephonic periodicities. From above this low periodicity we can, therefore, represent them with good approximation by the simple formula $a+bn$, where a and b are constants.

To arrive at the constants from my data I take the losses at 16 periods and deduct this from the losses at 40 periods, the remainder is then divided by 16, this gives us b; 16 times b is then deducted from the loss at 15 periods. The remainder is a.

Below are given the figures for the curves in the diagram calculated on the above basis and in the same units as before. For comparison with them I have also added the continuous or steady field losses on the same basis.

TABLE IV.—*Of Losses in Dielectrics.*
Values of constants a and b and steady field loss.

Substance and state.	Constant a.	Constant b.	Comparative steady field loss.
Blotting paper, room dry.	1,050,000	2,500	770,000
Celluloid " "	36,625	735	31,000
" dried 100°C.	2,450	100	340
Crown glass	700	137	28
Gutta percha	113	27·2	1
	Liquid.		
Thin mineral oil	13,150	56	4,200

Derivation of constants :—
From loss at 40 $\sim$.
Deduct loss at 8 $\sim$.
Difference $\div(40-8)=32=b$.
Loss at $8\sim-8b=a$.
Loss at any frequency above $8=a+bn$ approximately.

It will be seen that while at the lowest resistance the a constant does not differ much from the steady field loss, the difference rapidly gets greater as the resistance of the dielectric is higher till the two seem to have no relation, although the data given in the table show unmistakably that they have. From these figures it will be seen that the curves do not cross each other at higher frequencies unless the dielectric is abnormal, and that the above relations generally still hold good.

Another point not clearly seen in the curves, but which can be readily deduced from the tables, is the forms the curves assume near their origin. Although from the general reduction of scale the curves seem to become flatter as the resistance becomes higher, this is not so comparatively. If in each case the steady field loss is taken as unity and curves of the comparative losses are drawn for the different periodicities the rise and sweep of the curves is greater comparatively as the resistance is higher. There are, however, differences in detail which would repay further study. Heterogeneity or great strain also affects the lower or b part of the curve. I have got out comparative data for the Bohemian glass flask from 1 to 200 periods. In this extreme case the a constant is almost negligible, and the very simple formula bn fits fairly and to a first approximation the whole curve from 1 to 200 periods. Even a single degree's temperature difference also considerably affects the exact shape of the low part of the curves. The effects of temperature are, however, too complicated to be dealt with in this Paper, especially as they do not seem to affect the general conclusions to which the above experiments lead.

When it is remembered that in the origin of these curves we are covering from another aspect the phenomena which have been studied under the term absorption, and when it is also considered that this term also covers the complementary capacity effects, there is no need to wonder longer why the elaborate attempts of numerous workers to find some formula of a fairly simple character to express their results have not met with success, and must be looked on, in the present state of our knowledge, as highly empirical.

Liquid Dielectrics.

In their general features the actions in liquid dielectrics in electric fields are the same as in solids, as can be seen from the data given above (Table II.). There are, however, charac-

teristic differences in detail. In the first place the "Blotting paper" type of curve persists more with liquid dielectrics, that is, at low frequencies the curves tend to be more square-shouldered. Then, for a given specific resistance the ratio of the steady to the alternating losses is generally less than in the case of solids; this apparently is the most characteristic difference. I am sure, however, from my results that further detailed study would show other interesting deviations without upsetting the general resemblance in principle.

I have found, however, two difficulties in dealing with liquid dielectrics, especially the oils which make it less easy to get exact results than with solids. They very readily take up moisture from the atmosphere, which with dry oils alters their action while experiments are proceeding unless special precautions are taken against it. Secondly, there is another action when the oils have much moisture. I am then inclined to think that this moisture exists in the form of a fine emulsion in them. Such moisture existing then in globules is attracted to the electrodes wherever there is variation in the strength of the field. The result is that there is difficulty in getting a definite reading in steady fields; it diminishes for some time after the field is put on, and this is not, I think, ordinary absorption. This action also occurs to a less extent in alternating fields of low frequency. It is necessary also to be very particular about the cleanliness and dryness of all receptacles used. It is, however, quite easy to get results sufficiently near to show the general nature of the actions such as those given. It is only when one comes to more precise measurements and the reproduction of results that the above actions interfere.

Range of Actions.

The alternating losses in different dielectrics have a much smaller range than those due to equivalent steady pressures, notwithstanding that the two appear to be intimately connected. The alternating losses are in all cases the larger, but whereas at 40 ∞ and over the wide range of dielectrics included in the above tables :—

The smallest A.C. loss has a ratio to the largest of	1 :	1,100
The smallest C.C. loss has a ratio to the largest of	1 :	1,300,000

Or the range of the alternating-current losses is only about 1/1,200th that of the continuous-current range.

Reference to the tables and data given above will show further that the range of the alternating losses is still further

reduced at higher frequencies; in fact, to something like 1 : 300 over the above range before it becomes nearly constant.

This difference in the two ranges of losses brings forward the question as to what part of the ranges respectively it is which show the greatest differences. It is at the lower resistances or where the losses are greatest that the continuous-current and alternating-current losses are more nearly proportionate to each other, the difference becoming more accentuated as the resistances get higher or, as it is called, the dielectrics become more perfect. These interesting and suggestive differences are best studied by careful comparison of the figures given in the first two tables. When I first got out these relations I thought it looked as if some intrusive substance were present which by drying or otherwise could be got rid of, leaving a final action which must be on the substance itself, or be a true dielectric hysteresis, but subsequent consideration has led me to take another view.

To develop this a number of further experiments would need to be described, covering the temperature, capacity and pressure effects in steady and alternating fields and over a range of periodicities, and also experiments on the effects of moisture, in which I have endeavoured to get an idea of the amount of moisture which corresponds with given magnitudes of the actions occurring. The description of these experiments must, however, be reserved for another Paper or Papers.

I have to thank Mr. C. W. S. Crawley for much kind help and criticism in this work.

ABSTRACT.

After references to former work, especially on surface leakage, tables are given showing an intimate connection between the losses in steady and alternating fields, and that the one can be predicted from the other to a first approximation. Exceptions are mentioned, and it is shown that there is heterogeneity in these cases. Curves of the losses from 1 in 4 seconds to 40∿ are given for specimen dielectrics. Above 8 to 12∿ both for "good" and "poor" dielectrics, these become rising straight lines. The salient features lie below about 16∿.

The formula $a+bn$ applies to all dielectrics for frequencies above about 8 ∿, but not below. The a constant is always larger than and bears only a very indefinite relation to the steady voltage loss. Liquid dielectrics behave similarly to solid with certain differences, particularly that for a given resistance the ratio of the steady to the alternating loss is less than with solids. There is a great difference in the comparative ranges of the losses. For the dielectrics tabulated these vary in a steady field from 1 to 1,300,000, but the corresponding alternating losses vary only from 1 to 1,100. The difference is principally in the "good"

dielectrics, the alternating losses in which at 40 $\sim$ are vastly larger than these in a steady field, though they still appear to be connected with them.

DISCUSSION.

Mr. D. Owen (communicated remarks) considered that the author's tests supplied data of great interest, though their value would be greater if more details of actual readings were supplied. The results support the $a+bn$ formula for dielectric power loss already advanced by previous workers. They show also that a and b are probably unconnected. This being the case, the Author has adopted a wrong basis of calculation in arriving at his main point. It would be better to show the relation between a and a_0 the direct current power loss. The discrepancies noted in regard to a few materials (good insulators) might thus be found to disappear.

Mr. E. H. Rayner: The author mentions periodicities of one in four seconds. How are the voltage current and power measurements made under these conditions? Are the instruments so damped as to give a natural period of a minute or two? Even in such a case one would expect very inconvenient oscillations. Perhaps the author very greatly increases the inertia of the moving systems. Taking the continuous-current readings after the length of time required to become approximately steady is no doubt the best that can be done. It would afford much more information if intervals equal to a few periods of the corresponding alternating tests could be used. This would involve something of the nature of an oscillograph or an Einthoven galvanometer. As regards the curve 2 of diagram 1 I should have expected it to have been above curve 1 on account of the great resistance of guttapercha to continuous currents. The remarks on the shape of curves obtained with liquid dielectrics are interesting. I can confirm the effect of electric stress on oil. When damp the globules of water seem to appear from nowhere and grow and wriggle under the electric forces. Finally they may cause a discharge and are boiled away, and so in time they may disappear in stages and the liquid will sustain a higher voltage than it would at first.

Mr. A. Campbell (communicated): The careful observations recorded in the Paper are of value as widening our knowledge of the behaviour of dielectrics. It is interesting that in general there is no great change in the losses, as the frequency is lowered to 1 or 2 $\sim$ per second. A comparison of the steady current loss with the alternating appears to indicate that they have no clear connection. The practice, in the steady current test, of keeping the voltage applied until the apparent resistance has reached a steady value does not seem to give conditions comparable with those for the alternating voltage. Could the author, by means of his electrolytic commutator, go down to much lower frequencies, say, 1 $\sim$ in 100 seconds? By using a galvanometer of not too slow period and integrating the instantaneous product of the voltage and current curves the true alternating losses at such frequencies could be determined. The absorption effects in condensers certainly vary considerably over the first 100 seconds of charge. The existence of a state of strain in some of the glass tested, as suggested by the author, is rather improbable. This might be tested, however, with crossed nicols or the losses might be re-determined after re-annealing the glass.

Mr. G. L. Addenbrooke (communicated reply) expressed his obligation to those who had discussed his Paper for their suggestions. The difficulty was not to think of possible experiments but to select the most promising. He had not attached any physical meaning to the $a+bn$ formula. It was convenient to express the results over a certain range, but he thought a true solution would be found in some amplification of the formula employed by Ashton, and elaborated by other workers as far as possible with the data at their disposal. Such a final formula must include temperature coefficients, and will be of considerable complexity. In reply to Mr. Owen, the ratio of

a to a_0 is shown in the Paper for 40 $\sim$ for five dielectrics, but a more precise study of this part of the curve would be useful. In reply to Mr. Rayner, there are four impulses in one direction per period in the case of the voltmeter and two for the wattmeter, so that the swing even at 1$\sim$ in 4 seconds is not so great as might be expected. The instruments are also made nearly dead-beat by a system of air damping. In using instruments of the galvanometer type, as suggested by Mr. Rayner and Mr. Campbell, there is a difficulty with some dielectrics in separating the losses from the capacity current even at very low frequencies. By the term "strain" a difference in structure or state of aggregation was intended rather than any mechanical strain, but he hoped to deal further with this point later.

XXII. *Second Report to the Council of the Committee on Nomenclature and Symbols.*

THE Committee consists of Prof. Sir J. J. Thomson, O.M., F.R.S. (President), Prof. H. L. Callendar, F.R.S., A. Campbell, Esq., B.A., Dr. C. Chree, F.R.S., Prof. G. Carey Foster, F.R.S., Dr. W. Eccles (Secretary and Convener), Sir George Greenhill, F.R.S., Dr. Alexander Russell, M.A., Prof. the Hon. R. J. Strutt, F.R.S., Prof. S. P Thompson, F.R.S., Dr. W. Watson, F.R.S.

In agreement with the recommendations of the International Electrotechnical Commission, the International Commission on the Unification of Physico-Chemical Symbols, and other bodies, the Committee recommend :—

That Italic, not Roman, letters be used as symbols for the magnitudes of quantities in all branches of Physics. This applies to capitals as well as to lower-case letters.

The following list of symbols for quantities occurring in the subject of Heat was adopted :—

Quantity.	*Recommendation.*
Reciprocal of mechanical equivalent of thermal unit $(1/J)$	A
Constant in Van der Waal's formula $(V-b)(P-a/V^2)=RT$	a
"Covolume" in ditto	b
"Coaggregation" volume$=a/RT$, in approximate formula $V=RT/P+b-c$	c
Cooling effect of Joule-Thomson$=(dT/dP)$ at constant total heat	C
Intrinsic energy	E
Thermodynamic potential, $T\varphi-H$, or $H-T\varphi$ if preferred	G
Acceleration of gravity	g
Total heat $E+PV$ of vapour	H
Total heat of liquid	h
Mechanical equivalent of thermal unit in gravitational units (*e.g.*, foot-pounds)	J
Thermal conductivity	K
Diffusivity	k
Latent heat of vaporisation	L
Mass or molecular weight	M

Quantity.	*Recommendation.*
Indices in such expressions as VT^m, PV^n	m, n
Pressure	P
Vapour pressure of liquid or saturation pressure	p
Quantity of heat energy	Q
" Dryness fraction " or " quality " of mixture of liquid and vapour	q
Gas constant (per unit mass)	R
Specific heat of vapour at constant pressure ...	S
Specific heat of vapour at constant volume, and specific heat of liquid or solid	s
Temperature reckoned from absolute zero.....	T, θ
Temperature from freezing point or 0°C.	t, θ
Velocity	U, u
Specific volume of vapour or gas.....	V
" " liquid	v
Work	W, w
Entropy of vapour	Φ
" liquid	φ
Ratio of specific heats	γ or n
Density	ϱ
Efficiency.....	η

Nomenclature.

Recommended.—That the term calorie should not be used alone, but with a prefix indicating the unit of mass, the implied temperature scale being always Centigrade. *E.g.*, gram-calorie, kilogram-calorie, pound-calorie, which might be abbreviated to g.cal., kg.cal., lb.cal.

XXIII.—*The Change in Thermal Conductivity of Metals on Fusion.* *By* Alfred W. Porter, *D.Sc., F.R.S., and* F. Simeon, *B.Sc., Research Scholar University College, London.*

Received March 24, 1915.

It is well known that when fusion of a pure metal takes place there is, in most cases, a sudden drop in its electrical conductivity. For example, the electrical conductivity of liquid mercury is only about one-quarter of that of solid mercury at the same temperature. Exceptions are bismuth and gallium, for which the conductivity in the liquid state is double that in the solid. The object of the present investigation was to find whether there is a corresponding change in the value of the thermal conductivity. According to the simple form of the electronic theory usually given, the ratio of the two types of conductivity should be $\frac{4}{3}\left(\frac{a}{e}\right)^2 T$, where a and e are universal constants; and although we know that this law is obeyed only roughly, yet the correspondence is sufficiently near to show that the two phenomena are intimately connected.

The outcome of the investigation is to show that the change in the thermal conductivity on fusion is of the same order as that of the electrical.

Outline of Method.

The method of measurement consisted in the determination of the gradient of temperature in a column of the metal contained in a glass tube, and heated at one end to such an extent that half the column is molten, while the cooler half is solid. The metals employed were sodium and mercury. In both cases the column was placed vertically. In order to diminish convective loss of heat from the surface it was heated at the top in the case of sodium whose melting point is above atmospheric temperature, while in the case of mercury the lower half was frozen by insertion of the end of the containing tube in solid carbon dioxide. The apparatus used in the case of sodium is shown in Fig. 1. The heating vessel contained lead which could be melted by means of a flame and kept at a constant temperature. Into the base of the vessel was screwed a piece of brass rod about 1½ in. long and ⅜ in. diameter. This brass rod served to convey the heat to the sodium with which it came into contact. The glass tube which contained the

sodium was furnished with 12 tubular depressions which were made by locally melting the glass with a fine flame, and then pressing it in with a knitting needle. These tubular depressions were destined to receive the thermoelectric junctions by means of which the temperatures at the various points of the column of metal could be determined. Each junction consisted

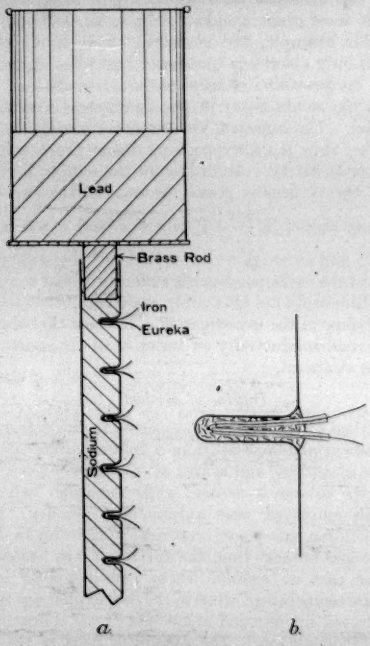


Fig. 1.

of fine iron and eureka wires (gauge 30-36). For the purposes of insulation fine glass tubes about 1½ in. long were slipped over the wires to separate them one from the other, and were then fused to the wires. Each junction so formed was inserted in a tubular depression, and the cavity was filled up with a packing

of tinfoil so that the presence of the junctions should make as little disturbance as possible in the thermal stream lines in the sodium (Fig. 1, *b*). The arrangement is shown in Fig. 1, *a* and *b*.

The wires were all of the same length, and were connected by copper wires (all of the same length as one another) to a moving-coil galvanometer of low resistance in series with a resistance-box for altering the sensitiveness. By making the

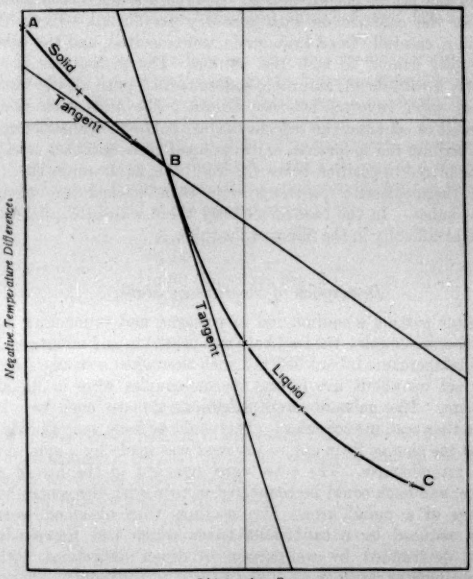


FIG. 2.—TEMPERATURE CURVES AND TANGENTS FOR MERCURY.

wires of equal length the temperature value of a scale division was made the same for all the couples. Since there was a resistance of 700 ohms in series with the galvanometer, any fine adjustment of these lengths was unnecessary.

The preparation of the sodium specimens was somewhat complicated by its solidity at ordinary temperatures, and its

oxidation on exposure to air. The following method was therefore adopted in preparing a bar. A glass tube, about 160 cms. long and 2 cms. diameter, was taken, and inset tubes made in its side as described above. About 10 cms. from the first of these tubes and 30 cms. from the nearer end, a constriction was made so that the end piece formed a funnel. A piston formed from another glass tube worked in from the other end. With the piston pushed well in, the tube was filled to a point above the constriction with pure dry paraffin oil. Pieces of sodium, carefully freed from oxide, were inserted, and the tube carefully heated to melt the sodium. The piston was then slowly drawn down, and more sodium added until all the inset tubes were covered by the sodium. The tube was then allowed to cool and the top cut off just below the constriction. The sodium rod so formed, in its casing of glass, was then ready to be placed in position below the lead bath, as shown in Fig. 1. The thermoelectric junctions were then packed into their inset tubes. In the case of mercury there was, of course, no special difficulty in the filling of the tube.

Description of Observations Made.

After putting a sodium rod in position, and connecting up the thermo-couples, the lead bath was heated to and maintained at a temperature (about 350°C.), such that when a steady state was set up about five of the thermo-couples were in liquid sodium. The galvanometer deflections due to each couple were then read in succession. This could be done very quickly, since the change from one to the next was made by a mercury cup interchanger. The cups were arranged in the arc of a circle, and each could be connected in turn with the centre by means of a radial arm. The readings thus obtained were then reduced by a calibration curve which had previously been determined by comparison of direct deflections with potentiometer readings.

These reduced readings were then plotted against the positions of the rod to which they correspond. To find from this curve the ratio of the conductivities of the liquid and solid, it is only necessary to find the two tangents which can be drawn at the point corresponding to the junction between the liquid and solid portions of the rod; because this ratio is inversely proportional to the ratio of the slopes of the two tangents.

In the case of mercury, arrangements had to be made to cool at the bottom instead of to heat at the top. The lower end was inserted into a carbonic acid snow and alcohol mixture. The whole tube was surrounded by a celluloid cylinder through which the thermo-electric couple wires were threaded. The object of this cylinder was to diminish the deposit of hoar frost in the lower part of the tube, which was otherwise very disturbing. The celluloid cylinder limited the rate of supply of fresh moist air to the tube, and thus achieved the desired object.

The change of slope is very much greater for mercury than for sodium. A typical curve for the former is shown in Fig. 2. The tangents at the point of change are also shown in this diagram. There is, of course, some difficulty in drawing in these tangents; hence, in different experiments considerable variation in slope was obtained. The following is a complete set of values for the ratio of the conductivity for the solid to that for the liquid at the same temperature :—

K_s/K_L for mercury.
4·44
4·62 (Curve on Fig. 2.)
3·28
3·76
3·83
3·53

3·91 = Mean K_s/K_L for mercury.

In the case of sodium it was found more satisfactory to plot log θ against x, because the slopes, especially that on the liquid side, were then more constant.

The following slopes were obtained :—

Sodium.

Solid.		Liquid.
0·0279	.....	0·0230
0·0267	.....	0·0188
0·0248	.....	0·01885
0·0267	.....	0·0209
0·0269	.....	0·0200
Sum 0·1330		0·10155

Ratio of slopes $\frac{0\cdot1330}{0\cdot10155} = 1\cdot31$ = mean K_s/K_L for sodium.

Electrical Conductivity.

The change of electrical conductivity with fusion had been determined by other observers both for mercury and sodium. The following determination for mercury are on record :—

Cailletet and Bouty (1885)	4·08
Dewar and Fleming (1896)	4·04
C. L. Weber (1885).....	4·10

In the case of sodium, previous determination of the change in electrical conductivity are :—

Matthiesson (1857)....	1·37—1·70
Bernini (1903).....	1·35
Northrup (1911).....	1·434 (Estimated from diagram.)

It was deemed advisable to make a fresh determination for sodium. A glass tube 0·5 cm. in diameter with two platinum wires sealed in at points about 5 cms. apart, was filled with sodium in a similar way to that described above. The tube was then set up in a horizontal electric furnace in such a way that the two platinum wires were near the centre of the furnace. Current was passed through the sodium, contact being made with it by stout copper electrodes. The potential-difference tapped off by the platinum wires was balanced on a potentiometer. No absolute measurements were made. Readings of the potential drop were taken at various temperatures below and above the melting point of sodium, and for both ascending and descending temperatures. The constancy of current through the sodium was checked by taking the potential drop along a piece of thick copper wire connected in series with it. The potentiometer readings were plotted against the temperature, and the ratio of the temperature coefficients for liquid and solid obtained by determining the slopes of the curves on the two sides of the melting point.

Four independent values obtained for the ratio were—

1·453
1·402
1·430
1·467

giving a mean of 1·438. This value is very similar to Northrup's, but is rather higher than our thermal values.

It may be of interest to mention some of the other changes which occur during fusion. There is, of course, a change in the specific energy (or internal latent heat)—that of the liquid being

the greater; a change in the density that of the liquid being less in the case both of sodium and mercury, but greater in the case of bismuth; a change in the coefficient of expansion that of the liquid being the greater; the specific heat of sodium undergoes no appreciable change, but its rate of variation with temperature changes from positive to negative on fusion (Ezer Griffiths, Roy. Soc. "Proc.," A., Vol. LXXXIX., 1913-14, p. 561); the temperature coefficient of the electrical resistance of mercury changes from 0·00455 between —55°C. and —40°C. to 0·000834 between 0°C. and 5°C.

With regard to thermo-electric effects the evidence is somewhat more conflicting. W. B. Burnie ("Phil. Mag.," 1897) finds a continuity in the curve of E.M.F. against temperature in the case of a copper-mercury element with, however, a marked change in its slope, $\frac{dE}{d\tau}$, at the melting point of mercury. The slope changes abruptly from a large nearly constant positive to a smaller negative value. This would, of course, imply a change in the Peltier coefficient $\left(\tau\frac{dE}{d\tau}\right)$. Assuming that the Peltier coefficient is proportional to the logarithm of the ratio of the concentrations of the electrons in copper and mercury, his values would give $n_{\text{solid}}/n_{\text{liquid}}$=about 5. This is sufficiently near to the ratio found for the electrical conductivities to tempt one to suppose that the two phenomena are telling consistent stories. On the other hand, Peddie and Shand ("Proc." Roy. Soc., Edin., 23, p. 15, 1900) find no change; nor also did P. Cermak ("Ann. der Phys.," 26, 1908, p. 521) for either the thermo-electric power or the Peltier effect.

Lastly, the photo-electric effect shows no change on fusion of the metal (sodium) according to Dember ("Ann. der Phys.," 23, 1907, p. 957). These results are discussed by E. Wagner ("Ann. der Phys.," 33, 1910, p. 1484).

The experimental part of this Paper was finished in July, 1913. In September, 1913, of the "Physical Review" appeared a Paper by J. W. Hornbeck on "Thermal and Electrical Conductivities of the Alkali Metals," in which is described a very similar method of investigation. In the case neither of sodium nor potassium was the investigation carried as high as the melting point; the thermal conductivity for both the solid and the liquid was determined for potassium-sodium alloy, the value being less in the liquid state. As the values were not obtained at the melting point itself, and as

there is only one value for the solid, it is not possible to state the exact ratio of reduction for this case.

The only other determination with which we are acquainted is one made by C. Barus ("Phil. Mag.," 33, p. 431 (1892), on thymol, a dielectric). He finds for the ratio of the thermal conductivity of solid to that of liquid the value 359/313. He had previously measured the change in specific heat. The corresponding change in thermometric conductivity is 1077/691.

ABSTRACT.

The change in question was determined for mercury and for sodium by finding the temperatures at different points of a cylinder of the metal contained in a glass tube. The ends of the cylinder were maintained at such temperatures that the metal was liquid half-way down its length, the remaining part being solid. The temperatures were taken by means of thermoelectric junctions inserted in narrow tubular depressions which had been formed in the glass tube by forcing a knitting needle down into the locally heated glass. The ratio of the thermal conductivity for solid and liquid was estimated from the slope of tangents drawn to the temperature-curve on each side of the melting point. The values of these ratios are of the same order as the ratio of the corresponding values of the electrical conductivities. The mean value for mercury is 3·91, and for sodium 1·31.

A summary list is given of other data concerned with fusion.

DISCUSSION.

Prof. O. W. Richardson congratulated the authors on their satisfactory treatment of an important problem. The change of thermal conductivity with fusion had not, he believed, been attacked experimentally before. In listening to the reading of the Paper one felt impressed by the small amount of effort and the simplicity of the apparatus with which the authors had been able to solve what was generally regarded as a very difficult problem. This research, it was interesting to note, was one of the few of which the results agreed with the deductions from the elementary electronic theory.

Prof. F. G. Donnan referred to the researches carried out at his suggestion by Dr. C. M. Stubbs on the radiation emitted by copper, silver and gold at high temperatures. These experiments showed that there was a marked change in the distribution of energy throughout the visible spectrum when passing from solid to molten metal, and pointed to some change in the electronic structure, or in the inter-electronic forces, occurring at the transition from solid to liquid. The complexity of the problem had, so far, prevented any mathematical treatment, but there appeared no doubt that results such as these, coupled with the discontinuities in thermal and electrical conductivity, would be of value in throwing light on the inner mechanism of the phenomenon of melting. Photometric measurements of the light radiated by polished plates of metals at high temperatures presented considerable difficulties owing to the tendency of the "flowed" surfaces to crystallise, but in spite of these difficulties Dr. Stubbs had clearly demonstrated the existence of an

abrupt change in the radiation-spectrum on passing from solid to liquid in the case of certain metals.

Dr. PORTER, in reply, said that while isolated measurements of thermal conductivity of some substances above and below the melting point had been made by previous workers, in no case that he was aware of had sufficient data been obtained from which to deduce the sudden change which takes place on fusion. He was cognisant of the work of Dr. Stubbs but had always thought the nature of the surface was of such importance in radiation experiments as to make it difficult, on account of crystallisation and contamination of the surface at high temperatures, to obtain conclusive results. He had forgotten at the moment the degree of consistency obtained by Dr. Stubbs, but if the results were more or less in agreement they would undoubtedly point to a change at the fusion point in the radiation emitted by the substances.

XXIV. *An Instrument for the Optical Delineation and Projection of Physical Curves.* By J. A. Fleming, *M.A., D.Sc., F.R.S.*

Received March 6, 1915.

It is often desirable and necessary to exhibit a physical curve such as a characteristic curve, hysteresis, or resonance curve in process of delineation, and to be able to record it by photography. The author has devised an apparatus which does this very conveniently, and can be applied in the delineation of physical curves of many kinds.

It consists of a base board on which is placed on suitable supports a rectangular rocking mirror pivotted on an axis parallel to one of the long sides of the rectangle. This mirror may be 1½ in. wide and 3 in. or 4 in. long. To the axis of the mirror is affixed a lever by which it can be turned through a certain limited angle. This lever is restrained by a spiral spring and pulled by a string which passes over a pulley. When the string is pulled the mirror is tilted about a horizontal central axis, and when the string is released the mirror returns to its original position. Over this mirror is another much larger one which is slung on and rotates round an axis at right angles to that of the first-named mirror, and serves to project on to a screen a ray of light reflected from the tilting mirror (*see* Fig. 1). On the same base board, which carries the supports of these mirrors, is fixed a thin wooden hoop like the rim of a wheel about 18 in. in diameter, 0·5 in. wide and 0·5 in. deep. This hoop is wound over with one layer of silk-covered Eureka, or high-resistance wire, in closely adjacent turns. The silk is rubbed off the top surface so as to expose the bare wire. This winding surrounds about three-quarters of the perimeter of the hoop. It is formed of No. 22 Eureka wire, and measures in resistance about 40 ohms. Three terminals are attached to it, one at the middle and one at each end. This wire forms a potentiometer wire, and a battery of 1 to 10 small secondary cells has its poles attached to the ends of this wire so as to make a fall of potential down it. In the centre of the hoop there is a pivot and a vertical axis with a radial metal arm, the end of which presses on, and moves over the bare exposed turns of wire on the hoop. A terminal is in connection with this axis. It will be seen that if the radial arm is moved either way from the central position in which it makes contact with the middle point of the potentiometer wire it takes off a certain

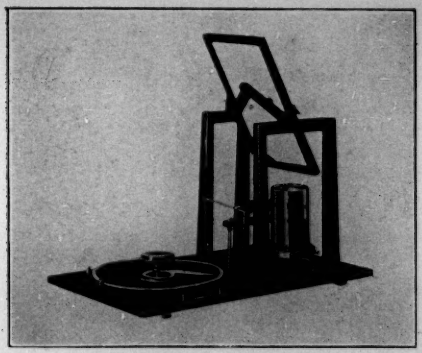


FIG. 1.—VIEW OF THE CURVE TRACER, SHOWING PROJECTION MIRROR AND CIRCULAR POTENTIOMETER.

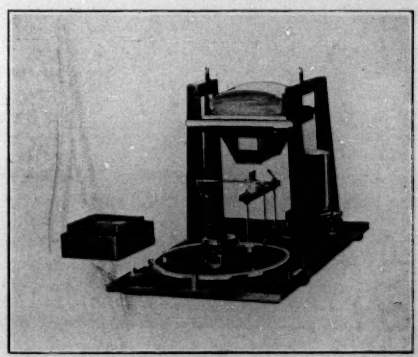


FIG. 2.—VIEW OF THE CURVE TRACER, SHOWING TRACING DESK AND PHOTOGRAPHIC CAMERA ATTACHMENT.

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fraction of the voltage of the battery, and we can create between the middle terminal of the potentiometer wire and the terminal in connection with the revolving arm any required fraction of the half battery voltage either in one direction or in the other. On the board is also placed a mirror galvanometer of movable coil or movable needle type or else a mirror magnetometer. A thin parallel beam of light is sent out from an electric arc lantern, and falls on the mirror of this galvanometer or magnetometer, and is thence reflected to the tilting mirror, and from that to the inclined projection mirror, and thence to a screen. The mirror galvanometer is so placed that right or left deflections of its coil or needle cause the ray to travel along the tilting mirror parallel to its long axis, and, therefore, the spot of light falling on the screen moves vertically in response to deflections of the galvanometer or magnetometer.

The thread attached to the lever fixed on the shaft of the tilting mirror is wound round a pulley on the shaft or axis of the radial arm, and so adjusted that when the radial arm is rotated the mirror is tilted through a corresponding angle. Hence the spot of light projected on to the screen has two motions—one, a horizontal motion which is proportional to the angle through which the radial arm is turned, which again is proportional to the P.D. created between the middle terminal of the potentiometer wire and the terminal in connection with the radial arm. Again, the spot of light can have a vertical motion imparted to it which is proportional to the deflection of the galvanometer coil or magnetometer needle, which, in turn, is proportional to the current through the galvanometer or to the deflecting magnetic field, which is at right angles to the direction of the needle itself.

Hence, if we cause the spot of light to be actuated by these two motions at once, it will describe a curve which will represent the relation between two variables, one proportional to the above mentioned voltage, and the other to the above mentioned current or field. Instead of receiving the reflected ray of light upon a screen it may be passed through a cylindrical lens placed over, and with axis parallel to, the tilting mirror, and may then be allowed to fall upon a photographic plate in a plate holder and record on it the curve in question (*see* Fig. 2).

As an example of one simple use of the above instrument, we may explain how it can be used to delineate and photograph a

magnetic hysteresis curve for iron. For this purpose a coil or helix is required wound with many turns of fine wire. The one used by the author is a coil wound with silk covered No. 36 S.W.G. copper wire in 37,363 turns and 21 layers. The wire is coiled on a brass tube 45·7 cm. long between the cheeks, and has a resistance of 979·3 ohms at 15°C. The value of the magnetic field per milliampere in the interior, near the centre, is approximately

$$H=\frac{4\pi}{10}\frac{N}{L}\frac{1}{1{,}000}=1{\cdot}021 \quad , \quad . \quad . \quad . \quad . \quad (1)$$

and the magnetic field per volt applied to the coil terminals is

$$H=\frac{4\pi}{10}\frac{N}{LR}=1{\cdot}05. \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Hence, the field is nearly 1 C.G.S. unit per volt impressed at terminals of the coil. Accordingly, by using a battery of 10 cells on the circular potentiometer wire and applying the tapped off voltage to the terminals of this coil we can create any magnetic force between —10 and +10 in the interior of the helix. If, then, we place on the curve tracer a mirror magnetometer and control the needle so that it stands with its axis parallel to the axis of the tilting mirror, and if we place the above-mentioned magnetising coil at some little distance away with its axis perpendicular to that of the magnetic needle and put in the helix an iron wire of length at least 200, or, better still, 400 times its diameter, we can apply to this wire a known magnetising force. It will then have created in it a certain magnetisation, and this will create an external magnetic field which will deflect the magnetometer needle through an angle which, if small, is proportional to the magnetisation and nearly to the flux density at the centre of the wire. Hence, if a ray of light is thrown on the mirror of the magnetometer, and reflected from the two mirrors of the curve tracer on to a screen we shall be able, by rotating the radial arm from one position on the potentiometer wire to another similar one on the other side of the middle point and then back again, to carry the wire through a magnetic cycle, and cause the spot of light on the screen to delineate a hysteresis curve or loop for any required range of magnetising force H between, say, +10 and —10. Moreover, we can calibrate the instrument so as to interpret the scale of the resulting curve and, therefore, calculate the energy dissipated per cycle of magnetisation (*see* Figs. 3 and 4).

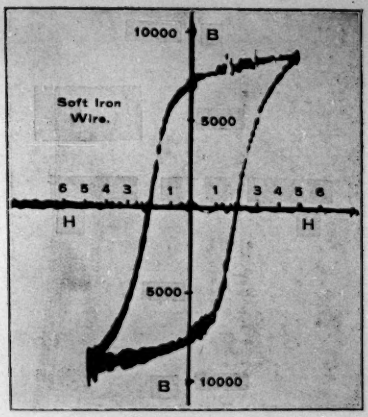

FIG. 3.—PHOTOGRAPH OF MAGNETIC HYSTERESIS CURVE OF AN IRON WIRE, TAKEN WITH THE CURVE TRACER.

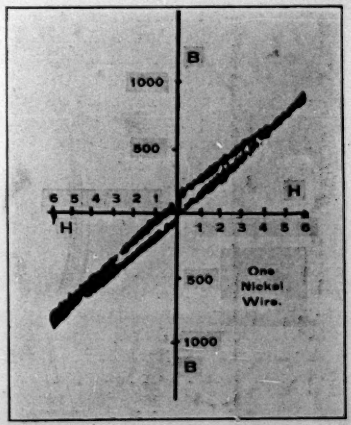

FIG. 4.—PHOTOGRAPH OF THE MAGNETIC HYSTERESIS LOOP OF NICKEL WIRE.

To face page 318.

To do this we replace in the exact position of the iron wire another helix or solenoid of equal length and known number of turns per centimetre, through which we can pass a measured current. Let this helix have N_1 turns per centimetre and let a current, A_1, be passed through it, which just makes the same magnetometer deflection as the iron wire at the extreme or maximum magnetisation. Then, $1{\cdot}25\,N_1A_1$ is the flux or field over the greater portion of the interior of the solenoid, and this must, therefore, be the same as the total flux in the iron-wire under the above conditions. Therefore, knowing the cross-section of the iron, we can find the flux density B in it, and hence the flux density scale for the hysteresis loop. Knowing the scale of the ordinates, we can then find the area of the hysteresis loop by marking out the path of the spot of light as it travels round one complete magnetic cycle.

If this path is marked off on paper we can cut out this curve and weigh the paper. We can then cut out of the same piece of paper a square of area known in units selected for B and H, and weigh that piece, and from the ratio of the weights find the area of the hysteresis loop in terms of that unit of area which is equal to the product of the lengths on the two axes which stand respectively for the unit of magnetising force H and the unit of magnetic flux density B. We know, then, that the value of $\frac{1}{4\pi}\int H dB$, or nearly 7/88ths of the area of the loop in this unit, gives us the work done in ergs in carrying unit volume of the iron round one complete magnetic cycle. Hence, we obtain the hysteresis for the maximum flux density employed, and, therefore, the hysteretic constant of the iron generally.

The calibrating coil employed by the author consisted of No. 22 silk-covered copper wire wound on a brass wire in 4,687 turns of eight layers. The length of the coil was 48·0 cm., and the diameter of the brass wire was 0·325 cm. diameter. Hence, the area of cross-section of the wire is 8·3 mm. square. The resistance of the coil was 5·73 ohms at 15°C. Hence, the magnetic force H in the centre is 122·5 units per ampere, or 21·4 per volt on the terminals. The cross-sectional area of the iron wires used in the above hysteresis measurement was 2·075 $(mm.)^2$, and hence the cross-section of the iron was 0·25 of the cross-section of the calibrating coil. The total flux embraced by the turns of this calibrating coil per ampere of current through it can be found by a single experiment with an embracing secondary coil and a ballistic galvanometer.

A second illustration of the use to which this instrument can be put is in the delineation of resonance curves.

In this case we replace the circular potentiometer wire by a variable air condenser of the multiple semicircular plate type, the capacity of which can be varied by rotating the insulated milled head which moves one set of plates. In the case of such a condenser the effective capacity is very nearly proportional to the angle through which the movable plates are turned, and can be made exactly by the use of a Seibt air condenser.

The string attached to the lever on the axis of the tilting mirror is then wound round the shaft of this condenser and adjusted so that rotating the movable plates pulls over the tilting mirror from one extreme limit of movement to the other. A suitable mirror galvanometer of low resistance is placed in position and connected to a thermocouple, pressing on a short length of fine high-resistance wire, which is inserted in the circuit of the variable condenser, and the said circuit is completed by a coil of suitable inductance, L, so that the oscillation constant of the circuit ($=\sqrt{CL}$) has a value near to that of the circuit to be tested. Let it be then desired to obtain the resonance curve of a wireless telegraph spark transmitter or of any other oscillation circuit. The above described condenser circuit is loosely coupled with the circuit under test, and a ray of light from an arc allowed to fall on the galvanometer mirror and to be reflected thence to the tilting mirror and projection mirror and then on to a screen. It is obvious that if the rotating handle or head of the variable condenser is slowly turned from one extreme position to the other, the circuit will gradually be brought up into resonance with the circuit under test, and then pass beyond it. The spot of light on the screen will, therefore, be caused to move horizontally by the change in capacity by a distance proportional to that change, and vertically by a distance which is approximately proportional to the square of the current in the condenser circuit. Hence, the spot describes the well-known path of a resonance curve which is more or less peaked in proportion as the total decrement of the two circuits is less. A few simple measurements suffice to calibrate or show the scale of the horizontal and vertical movements.

Taking, then, the well-known modification of Bjerknes' decrement formula applicable in this case, and calling C_r the capacity of the condenser setting corresponding to resonance,

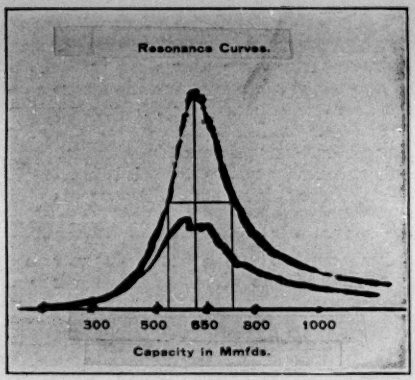

FIG. 5.—RESONANCE CURVES PHOTOGRAPHED WITH CURVE TRACER.

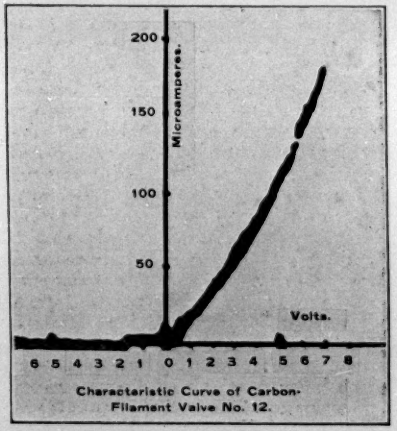

FIG. 6.—PHOTOGRAPH OF CHARACTERISTIC CURVE OF FLEMING OSCILLATION VALVE.

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and I_r the resonance or maximum current in the condenser circuit, whilst C and I stand for the capacity and current corresponding to any other setting, we have, provided $(C_r - C)/C_r$ is a quantity not exceeding, say, 5 per cent., the expression

$$D = \frac{\pi}{C_r}(C_r - C)\sqrt{\frac{I^2}{I_r^2 - I^2}}, \quad \ldots \ldots \quad (3)$$

where D is the sum of the decrements per complete period of the circuit under test and of the circuit of the curve tracer. Now, if we take such a value of the capacity C as to make the current

$$I = \frac{1}{\sqrt{2}} I_r = 0{\cdot}707\, I_r, \quad \ldots \ldots \quad (4)$$

then the quantity under the $\sqrt{}$ in the above formula (3) becomes unity, and we have

$$D = \frac{\pi}{C_r}(C_r - C). \quad \ldots \ldots \quad (5)$$

We can once for all find the value of C_r from the calibration curve of the condenser, and therefore the value of π/C_r, and therefore it is clear that the total decrement is proportional to the horizontal shift of the spot of light from the zero position which corresponds to a reduction in the maximum value of the mean-square current to half its value. Since the ordinates of the resonance curve will be nearly proportional to the mean-square current we have merely to ascertain the horizontal shift of the spot of light from its position corresponding to the maximum ordinate of the resonance curve when that ordinate is reduced to half its value.

It is quite easy by known methods to find once for all the decrement of the condenser circuit of the curve tracer, and, therefore, by subtraction, to find the decrement of the oscillation circuit under test.

The resonance curve can be photographed by projecting the spot of light on a slow photographic plate or on Eastman's bromide paper, by causing it to move slowly over its path once or twice. We have, therefore, in this appliance a very simple method of quickly taking a resonance curve and recording it (*see* Fig. 5).

By inserting in one or other of the oscillation circuits a piece of high-resistance wire, we can show how much the sharpness

of the decrement curve is reduced by increasing the damping in either circuit.

An interesting proof can be given of the co-existence of oscillations of two different frequencies in the circuits when they are closly coupled by the deformation of the resonance curve and the appearance on it of two humps.

We can, in the third place, employ this curve tracer to delineate the *characteristic curve* of any conductor which has a unilateral or unsymmetrical conductivity, or which in common parlance does not obey Ohm's law.

Thus, for instance, we may employ the variable P.D. tapped off the circular potentiometer to send a current through a junction of zincite and chalcopyrite, or of carborundum or any of the rectifying crystals or contacts employed as receivers in wireless telegraphy.

The current through this conductor can then be passed through the galvanometer of the curve tracer. The spot of light will then be actuated by two movements, one horizontal proportional to the applied voltage and one vertical proportional to the resulting current through the conductor. The spot of light then defines the characteristic curves. A number of curves of this kind delineated for various pairs of contacts were recently shown to the Physical Society by Mr. A. F. Hallimond in a Paper on "The Conduction of Electricity at Point Contacts." * He mentioned without detailed description that a form of mirror galvanometer was employed to delineate the curves, but I am not aware whether the arrangement was identical with this instrument now shown by me. This curve tracer can also be employed to investigate the rectifying action contacts or of my glow lamp oscillation valves used as detectors in wireless telegraphy, as shown by the reproductions of photographic curves in Figs. 6 and 7.

The above three illustrations do not by any means, however, exhaust the uses of this curve tracer. We can employ it to delineate any kind of cyclical curve in which one variable can be represented by and be proportional to the angle through which the lower rectangular mirror is tilted, and the other co-ordinate can be represented by the deflection of a mirror galvanometer, magnetometer, voltmeter or electrometer.

We may thus employ it to delineate temperature curves when there is temperature hysteresis in any cycle of operations

* *See* "Proc." Phys. Soc., London, Vol. XXVII., p. 70, 1914.

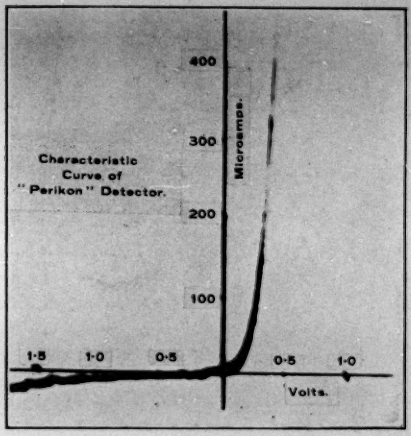

FIG. 7.—PHOTOGRAPH OF CHARACTERISTIC CURVE OF ZINCITE-CHALCOPYRITE DETECTOR.

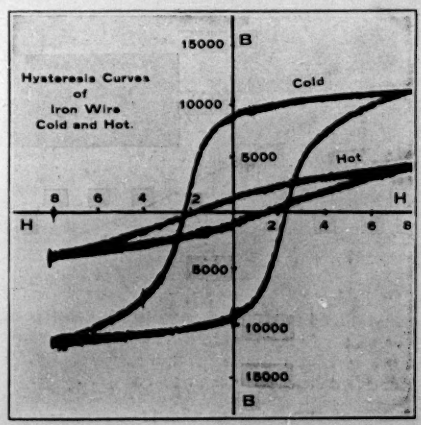

FIG. 8.—PHOTOGRAPHS TAKEN WITH THE CURVE TRACER SHOWING THE EFFECT OF HEAT ON THE MAGNETIC HYSTERESIS OF IRON.

To face page 322.]

or the relation of pressure and volume, expansion and temperature in cyclical operations.

In magnetic investigations it is sure to prove useful, because we can operate with the magnetised wire under test not only in a horizontal position, but in a vertical one. We can thus investigate the effect of vibration, torsion, temperature and electric oscillations on the hysteresis of iron and other magnetic metals, and see in a moment whether any particular physical or electrical operation increases or diminishes the magnetic hysteresis (*see* Fig. 8). The fact that the magnetic samples are used in the form of wires or narrow strips cut off a sheet is a great advantage and very much more simple than when rings have to be employed. If the wires employed are about 1 mm. in diameter and 400 mm. long they are in effect magnetically equivalent to rings. We can use this curve tracer to investigate in this manner samples cut from the actual sheet steel to be employed in making the core of a transformer or dynamo armature.

It is also easy to use the instrument to exhibit the recalescence points of iron by heating a wire by an electric current to bright redness, and then allowing it to cool, and causing the temperature to be measured by a thermo-electric junction, whilst the tilting mirror is slowly moved over by a clock or mechanism which records the time. The instrument exhibited has been made in the Pender Electrical Laboratory by my assistants, Messrs. P. R. Coursey and Williment, and Mr. Coursey has rendered me skilful assistance in taking the photographs exhibited of various physical curves.

Note Added April 26, 1915.

The possibilities of the above described instrument must not be altogether judged by the illustrations given in the above Paper from photographs taken with it. In the instrument actually constructed the optical arrangements were by no means as good as they can be made. By the use of properly worked galvanometer or magnetometer mirrors, and the adoption of well-known means for securing a smaller well-defined spot of light on the plate, the photographic curve could be made much sharper and uniform, and the definition far better.

Also, the author has, since the reading of the Paper designed another arrangement for the production of a uniformly varying

current by the rotation of an axis as follows : A ring is formed of high-resistance alloy which should be in the form of a strip or wire capable of carrying a current of 30-50 amperes. This ring has two electrodes attached to it at opposite ends of a diameter so that a current may be passed through the two halves in parallel by means of one or more large storage cells. A bar of wood or other insulating material is pivoted on an axis which is fixed at the centre of the ring and this axis has one end of the string attached to the tilting mirror fastened to it. The axis must be of such a diameter that when turned through one semi-revolution the string pulls over the tilting mirror through the entire range of motion permissible. The rotating diametral arm carries a pair of brushes or rubbing contacts which press on the two opposite sides of the ring, and these brushes are connected by flexible connectors or by rubbing contacts with two fixed terminals to which are attached the ends of the circuit in which it is desired to produce a continuously varied current in one or other direction.

It will be evident that when the diametral arm contacts are turned so that they make contact with two points on the ring 90 deg. removed from the two electrodes of the ring, then no current flows through the above circuit. If, however, the diametral arm is turned so that it connects the two electrodes of the ring then the current in the circuit is a maximum one way or the other. Hence by slowly moving this arm through 180 deg. and back we can carry the current in the circuit through a complete cycle and if this circuit includes a magnetising coil we can apply to an iron core within it, a cyclical magnetising force. This ring and its diametral arm would then take the place of the circular potentiometer in those cases in which it is desired to make the horizontal movement of the spot of light represent a larger current than can be produced by the circular potentiometer. The other arrangements would remain the same. It will, however, in general be found most convenient to retain the potentiometer method described in the Paper.

In the above described current-producing arrangement, there is, of course, a considerable waste of current which flows round the two sides of the ring and is generally large compared with that which flows through the diametral circuit. If the radius of the circular ring of resistance wire is denoted by r, and if ρ is the resistance per unit of length of this wire, and if R is the resistance of the radial arm and any coil included in it,

then it is easy to show that the current flowing through this radial arm is

$$x=\frac{V(\pi-2\theta)}{\pi R+2\theta(\pi-\theta)r},$$

where θ is the angular distance of the radial arm from the two diametral points on the ring at which the current enters and leaves, and V is their P.D. For if $\theta=0$ then $x=V/R$, and if $\theta=\pi/2$ then $x=0$, as it should be. Also if R is large then x is nearly proportional to $\pi/2-\theta$.

In the discussion which followed the exhibition of this instrument to the Physical Society, Prof. Silvanus Thompson asked if the instrument had received a name. Acting on a suggestion kindly given by Prof. Arthur Platt the writer proposes as the name for this instrument the *Campograph*, from Καμπη, a curve.

ABSTRACT.

This instrument is designed for projecting on to a screen or photographing on a plate such curves as magnetic hysteresis, resonance, or characteristic curves which can be performed slowly, or are non-periodic or non-repetitive. It consists of a base board on which is fixed a thin wooden ring wound over with one layer in close turns of insulated Eureka wire. The insulation on the top surface is removed so as to expose the wire. At the centre of the ring there is a pivot which carries a radial arm the end of which rubs on the bare Eureka wire. The ends of the Eureka wire are connected to a battery of n cells and the middle point of the wire is connected to a terminal, and also the centre of the radial arm to another terminal. The arrangement forms a circular potentiometer so that any required fraction of the battery voltage can be produced in either direction between the last-mentioned two terminals. On the same base there is a rectangular mirror slung on a longitudinal axis, which mirror is tilted by a thread wound round the shaft of the radial arm of the potentiometer. Hence, if a ray of light from a mirror magnetometer or galvanometer is reflected to the tilting mirror and thence on to another fixed mirror with its axis at right angles to that of the other and thence to a screen, the spot of light will have two motions, one a horizontal one proportional to the displacement of the radial arm, and, therefore, to the P.D. of the two terminals, and another vertical motion proportional to the deflection of the galvanometer or magnetometer. If, then, we place behind the magnetometer a long coil having a long iron wire within it and send a current through this coil which is supplied from the potentiometer travelling terminals, the spot of light will be actuated by two motions, one a horizontal one proportional to the magnetising force, and the other a vertical one proportional to the central flux density in the iron. It will, therefore, describe a hysteresis curve when the radial arm is moved to and fro cyclically through a certain arc or angle. In the same manner the

instrument can be used to delineate characteristic curves of wireless detectors or rectifiers or other devices. By using a rotating condenser of variable capacity in place of the circular potentiometer, the instrument can be used for delineating resonance curves of wireless telegraph plants.

DISCUSSION.

Prof. S. P. THOMPSON complimented the author on the simplicity and effectiveness of his apparatus in which, as was characteristic of Prof. Fleming's devices, were embodied the results of much careful thought and painstaking experiment. He himself had often felt the desirability of having an arrangement by which one could vary the magnitude of an electric current in such a way that it should be directly proportional to some movement in a straight line. He had thought out various potentiometers in this connection, but the difficulty with potentiometers was the small current obtainable from them which was a disadvantage in magnetic work. Prof. Fleming had surmounted the difficulty of straight line proportionality by working round an axle instead.

Mr. W. DUDDELL joined in congratulating the author on the wide utility of his instrument. He recalled that when delivering his first lecture on the oscillograph he had been confronted with the problem of projecting the motion of a spot of light in two co-ordinates on a screen and had used a very similar arrangement to Prof. Fleming's. He also had a circular rheostat, but instead of a string connection to the mirror he had used a cam. This apparatus had worked very satisfactorily.

Prof. E. WILSON said that the apparatus was undoubtedly admirable for demonstration purposes, and one naturally wondered what degree of precision would be possible with it if applied to research. It appeared from the step and stair motion of the spot on the screen that the speed would be limited on account of the damping of the magnetometer or galvanometer employed.

Prof. S. W. J. SMITH described a method which he had found to be simple and effective for tracing hysteresis loops. An ingenious rheostat had been devised by Prof. Morris in which by the rotation of a handle two contacts are made to slide in opposite directions along a resistance which is connected directly across the battery. The voltage at the contacts varies continuously between $\pm$ the whole P.D. of the battery, being zero when they are at the middle. This rheostat is mounted vertically, and a white card of suitable size is attached to one of the moving contacts from which the magnetising current is taken. Then if the magnetometer spot is focussed on the card and its position marked as the current is varied the hysteresis loop is easily traced out. The Morris resistance satisfies some at least of the requirements mentioned by Prof. Thompson, though for the method he had described simpler rheostats of similar type could also be used.

Prof. MORRIS agreed that the question of getting a current which is proportional to some linear motion was a difficult one. Before arriving at the design of the rheostat mentioned by Dr. Smith he had tried various other types, including a circular one similar to Dr. Fleming's. The main point about his rheostat was the magnitude of the currents which could be taken from it. It would stand 20 amperes for a short time.

Mr. A. F. HALLIMOND: As the author of the Paper mentioned by Prof. Fleming, he might perhaps explain very briefly the nature of the apparatus referred to. A slide which he exhibited showed a mirror galvanometer with one needle, which was freely suspended at the centre of the two coils, which were at right angles to one another. The beam of light from a small 4-volt lamp was focussed by a lens, and, after reflection from a thin glass plate, inclined at 45 deg. to the axis of the needle, travelled horizontally along the common diameter of the coils to the needle. A mirror attached to the needle re-

versed this beam, which came to a sharp focus on a screen at a distance of about 2 ft. Since writing this account his attention had been called to the curve tracer designed by Dr. Searle. In principle, he believed that the two pieces of apparatus were very similar. The assumption underlying Prof. Fleming's apparatus, that the voltage was proportional to the length of the potentiometer wire, was only accurate if the crystal had a very high resistance; this difficulty was completely avoided by the use of the other type of apparatus.

Prof. FLEMING, in reply, thanked the various speakers for their remarks and said that he hoped to improve considerably on the apparatus especially as regards the optical arrangements. When this had been done he thought it would be extremely accurate and quite suitable for research work.

XXV. *The Stability of Some Liquid Films.* *By* P. Phillips, *D.Sc., and* J. Rose-Innes, *M.A.*

Received March 23, 1915.

Many teachers of physics must have felt that the literature at present published on the theory of surface tension does not meet their requirements when they are engaged in lecturing on the subject to students. Such a book as Poincaré's "Capillarité" assumes considerable attainments in the reader; while the so-called proofs in Maxwell's article in the "Encyclopædia Britannica" and in text-books which employ easier mathematics are often wanting in rigour. The present Paper is an attempt to supply this deficiency; to indicate, in fact, how we can prove some of the interesting theorems in the subject by methods at once simple, rigorous and not too clumsy. We may begin by considering the form of an equilibrium surface which is symmetrical about an axis. The best and best known investigation on the subject is probably that given by Maxwell in the article in the "Encyclopædia Britannica." Maxwell's logic on this point is unexceptionable, but pedagogically his work is marred by two minor blemishes: (1) He introduces the notion of a rolling curve abruptly without his previous work having in any way led up to the idea; and (2) he assumes more knowledge of conics than is necessary. We believe that a slight deviation from his proof enables us to get rid of these blemishes.

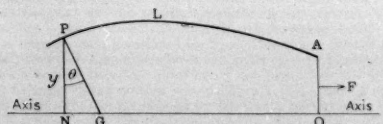

Fig. 1.

Consider the equilibrium of the portion of the film whose surface is produced by the rotation of ALP round the axis ON. AO is the radius of the end of the film which is kept stationary by having a force, F, applied to it parallel to the axis. PG is the normal to the surface and PN the normal to the axis, and

the angle NPG is θ. Then, if p is the excess of pressure inside the film over that outside, and T the surface tension,

$$F+p\pi y^2=2\pi y 2T\cos\theta,*$$

or

$$\text{Cos}\,\theta=\frac{p}{4T}\left(y+\frac{F}{p\pi y}\right).$$

To simplify the form of the equation write b^2 for $F/p\pi$ and a for $2T/p$. Then we obtain

$$\text{Cos}\,\theta=\frac{y+\frac{b^2}{y}}{2a}.$$

To obtain the form of the curve from this equation produce PN (Fig. 2) for a distance $NL=\frac{b^2}{y}$, and produce PG until it

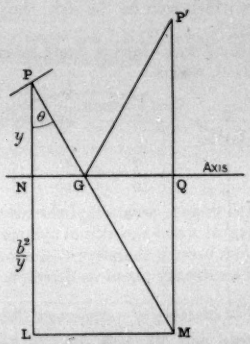

Fig. 2.

meets in M the line drawn through L parallel to the axis. Then PM is evidently equal to $2a$.

Draw MQP′ perpendicular to the axis making QP′ equal to QM, and join GP′. Then GP+GP′ is also equal to $2a$. Also the distance PP′ is given by

$$PP'^2=PM^2+MP'^2-2PM\,.\,MP'\cos\theta,$$
$$=4a^2-4b^2.$$

* This equation may also be derived from the second order differential equation, $P=2T\left(\frac{1}{R_1}+\frac{1}{R_2}\right)$ by one integration.

Therefore, as y and θ vary from point to point on the curve, G lies on an ellipse whose foci are P and P′, and whose major and minor axes are respectively $2a$ and $2b$. Since the two angles PGN and P′GQ are equal, the axis is a tangent to the ellipse, and since PG is a normal to the curve, the whole curve is described by P when the ellipse rolls on the axis.

The pressure within the film being $2T/a$, it depends on the major axis only of the ellipse.

If the pressure within the film is less than outside, p is negative, and it can be shown by a method similar to the above that the curve becomes the roulette of one focus of a hyperbola.

If the pressure inside the film is the same as outside, p is naturally zero and the equation becomes

$$y \cos \theta = 2a.$$

This can be shown just as simply to be the roulette of the focus of a parabola which rolls on the axis, the latus rectum of the parabola being $8a$.

By substituting dy/dx for $\tan \theta$ and integrating, and making $x=0$, when $\theta=0$, we get

$$\frac{y}{2a} = \cosh \frac{x}{2a},$$

or, putting c for $2a$,

$$\frac{y}{c} = \cosh \frac{x}{c}.$$

This is the well-known equation of the catenary with its directrix as the axis of x and the axis of symmetry as the axis of y. The equilibrium form is therefore a catenoid generated by the revolution of a catenary about its directrix.

The Stability of a Spherical Bubble.

Both Plateau and Maxwell seem to have been somewhat careless in stating the conditions of their films when they investigated the stability. The necessity of stating the conditions exactly is illustrated by means of the spherical soap bubble. A bubble of radius a and surface tension T will be in equilibrium with a source of gas kept at a constant pressure $4T/a$ above the atmospheric pressure. If the bubble expands, its equilibrium pressure* is reduced, and the source will force

* The term "equilibrium pressure" is used to denote the pressure with which the film would be in equilibrium, and the term "neighbouring equilibrium form" will be used to denote the neighbouring equilibrium form having the same equilibrium pressure.

more gas into it and expand it still further. Similarly any small shrinkage of the bubble will go on increasing. The bubble is thus always unstable at constant pressure.

For a completely closed bubble we must introduce Boyle's law as a condition. The equilibrium pressure to start with will be $4T/a$. If, therefore, the radius changes by a small amount to a_1, the equilibrium pressure becomes a/a_1 times as great, while the actual gas pressure becomes a^3/a_1^3 times as great. Thus, the change in gas pressure more than compensates for the change in equilibrium pressure, and for this type of displacement at any rate the bubble is stable.

The Stability of a Cylindrical Film.

This may also be considered under the same two conditions, first, at constant pressure, and, second, when the film encloses a constant mass of gas.

1. *At Constant Pressure.*—Let LM, NP represent the section of a cylindrical film whose axis is QR.

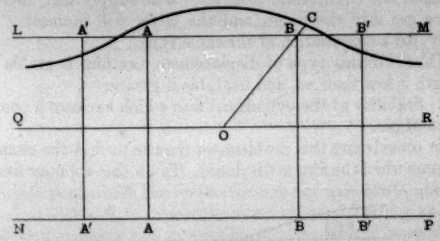

FIG. 3.

Then LM is the roulette of a focus of an ellipse which has become a circle—*i.e.*, of an ellipse which has zero eccentricity and whose semi-major axis is equal to the radius a, of the cylinder. The neighbouring equilibrium form will be generated by an ellipse of small eccentricity and with the same major axis. Since this ellipse is nearly a circle of radius a, the points where this neighbouring form crosses LM are πa apart in the limit. Let us suppose that the two fixed ends AA, BB of the film are less than πa apart, and that a bulge of the neighbouring equilibrium form is placed symmetrically between AA and BB.

This bulge will evidently pass outside AA and BB. In order to make it pass through them we may shrink it radially towards the centre O of the film in the ratio OB/OC, thus producing a curve which is described by an ellipse of the same eccentricity, but of smaller major and minor axes.

The equilibrium pressure of this last form will be greater than that of the undistorted cylinder, and, therefore, greater than that of the constant pressure supply. The film will, therefore, force gas back into the supply and straighten itself out into the cylindrical form.

A similar argument shows that a small constriction of the same form also straightens itself into the cylindrical form.

If the distance between the ends is greater than πa (A′B′ in the figure), and a bulge of the neighbouring equilibrium form be placed symmetrically between A′ and B′, the curve will pass inside A′ and B′. We may make it pass through these points by expanding it radially, but in so doing we shall have increased the major axis of the generating ellipse, and so reduced the equilibrium pressure. The supply will, therefore, force gas into the bulge, and the latter will increase. Similarly, for a constriction of the same type.

Thus, for this type of displacement the film is stable if its length is less than πa, and unstable if greater.

2. Stability of the cylindrical film which encloses a constant mass of gas.

In considering this problem we require to find the change in volume when the film is displaced. To do this we may first de-

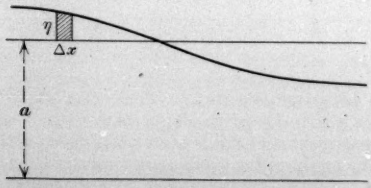

Fig. 4.

termine the change in area of the generating curve when the surface is altered from the cylinder to the neighbouring equilibrium form. The radius of the cylinder being a (Fig. 4), the ordinate at any point of the neighbouring curve may be written $a+\eta$

where η is small. Then the element of the change of area is $\eta\Delta x$, adopting the usual notation. Now, since the loss of curvature due to the change of ordinate from a to $a+\eta$ is compensated by the gain due to curvature of the generating curve, we must have

$$\frac{1}{a}-\frac{1}{a+\eta}=\frac{1}{R},$$

where R is the radius of curvature. This equation gives

$$\frac{\eta}{a(a+\eta)}=\frac{1}{R}, \text{ or } \eta=\frac{a^2}{R} \text{ approximately.}$$

Hence,
$$\eta\,.\,\Delta x=a^2\frac{\Delta x}{R},$$

and $\Delta x/R$ is easily seen to be equal to $\Delta\theta$, neglecting second order quantities, where θ is the inclination of the curve to the axis.

Then, $\eta\,.\,\Delta x=a^2\,.\,\Delta\theta$, and the total change in area between

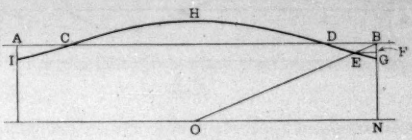

FIG 5.

two points at which the inclinations to the axis are θ_1 and θ_2 is equal to $a^2(\theta_2-\theta_1)$. Since the same value of θ recurs at intervals of $2\pi a$ along the curve, the total change of area will be zero when the length is equal to $2\pi a$.

Again, neglecting quantities of the second order, it follows immediately from this that the volume of the neighbouring equilibrium form of length $2\pi a$ is the same as the volume of the cylinder.

If the film has a length smaller than πa, a symmetrical bulge of the same type as considered before will evidently result in an increase of volume, and, therefore, a decrease of gas pressure, as well as the increase in equilibrium pressure; the tendency for such a bulge to straighten out into the cylindrical form will, therefore, be stronger than in the constant pressure case.

If the length (AB in Fig. 5) is between πa and $2\pi a$, the

volume enclosed by the neighbouring equilibrium form with a bulge symmetrically placed between A and B is greater than that of the cylinder, since the area of the sector CHD is greater than the sum of the portions, each less than half a sector, ACI and DBG.

Expanding the curve radially from O in the ratio $\frac{OB}{OE}\left(=\frac{a'}{a}\right)$, we expand all the ordinates in the same ratio, for $\frac{OB}{OE}=\frac{NB}{NF}=\frac{NB}{NG}$, neglecting the second order quantity FG.

The volume of the film thus displaced will be more than $\left(\frac{a'}{a}\right)^2$ times the volume of the cylinder, and thus the gas pressure will be less than $\left(\frac{a}{a'}\right)^2$ times the original pressure. Since the equilibrium pressure is only decreased in the ratio a/a' times the original, the change in gas pressure more than compensates it.

Fig 6.

A similar argument applies to a constriction of similar form, and thus the film will be stable for this type of displacement up to and evidently beyond a length $2\pi a$. At the length $2\pi a$ (AB in Fig. 6) an unsymmetrical displacement into a neighbouring equilibrium form becomes possible, as shown in the figure, with A and B still on the new curve. The volume enclosed by this displaced film being evidently unchanged, no additional protection is given by Boyle's law, and since both gas pressure and equilibrium pressure are unchanged, the film is evidently in neutral equilibrium as regards small quantities of the first order. If the length (CD in Fig. 6) is greater than $2\pi a$ we may imagine a new displaced form by expanding the neighbouring equilibrium form in the ratio $\frac{CD}{AB}(=\lambda$, say). This decreases the value of dy/dx at points having the same value of y in the ratio $1/\lambda$, and the value of d^2y/dx^2 in the ratio $1/\lambda^2$. Neglecting second order quantities, therefore, the curvature in the plane of the diagram is reduced in this ratio.

In the neighbouring equilibrium form the curvature in the plane of the diagram just compensates for the change in the value of y from the cylindrical form and therefore in the new displaced form this compensation will be incomplete. The displacement at each point will therefore increase.

If the length of the film is less than $2\pi a$, so that a longitudinal contraction is necessary, the curvature in the plane of the diagram will evidently be increased by the contraction, and will more than compensate for the change in the value of y. The displaced film will, therefore, straighten out into the cylindrical form.

Thus, for this type of displacement the critical length is equal to $2\pi a$—*i.e.*, the unsymmetrical type is a more severe test of stability than the symmetrical.

Since this unsymmetrical displacement involves no change in volume, the argument would apply equally whether the film is closed or connected to a source of gas at constant pressure. For the constant pressure case, therefore, the symmetrical displacement proves to be the more severe test of stability.

The Stability of a Film in the Form of a Catenoid with Equal Ends.

We have seen that the equation to this equilibrium form is $\frac{y}{c}=\cosh\frac{x}{c}$, when the axis of the film is the axis of x and the origin is at the centre of the film. The minimum value of y is evidently equal to c, and this is, therefore, the radius of the "neck" of the film. The tangents from the origin to the catenary are given by the condition $\frac{dy}{dx}=\frac{y}{x}$ at the points of contact with the tangents.

$$\therefore \operatorname{Sinh}\frac{x}{c}=\frac{c}{x}\cosh\frac{x}{c},$$

i.e.,
$$\tanh\frac{x}{c}=\frac{c}{x}.$$

This gives $\frac{x}{c}=\pm 1{\cdot}1996787$, and therefore we get $\frac{y}{x}=\pm 1{\cdot}507879$, and $\frac{y}{c}=\pm 1{\cdot}808971$.

From the form of the equation we see that the whole family of curves for different values of c can be derived by simple radial expansion or reduction of one of them. The tangents from the origin are, therefore, envelopes of the curves, and, of course, two neighbouring curves will cross at their points of contact with the envelope.

If A and B (Fig. 7) represent the trace of the fixed ends of the film we see that two of the catenaries can pass through them; one of these touches the envelope below A and B and the other above.

Let us consider the stability of the latter by assuming the following type of displacement. First, construct a neighbouring catenary of the same type but with slightly smaller

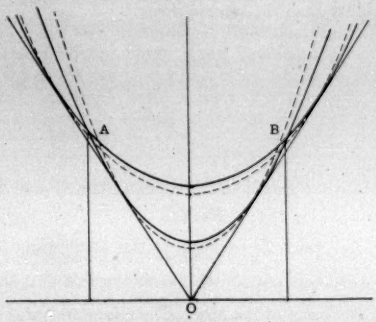

Fig. 7.

parameter c. Since this crosses the original catenary at the points of contact with the envelopes it will pass underneath A and B. Therefore, displace the whole curve upwards until it passes through A and B.

Each point on the new curve will still be displaced downwards between A and B, for the difference between the ordinates of two neighbouring catenaries having the same directrix increases as we pass from their common points to the neck.

After the first step in the displacement the total curvature is still zero, but by the second step the curvature in the plane perpendicular to the plane of the paper is everywhere decreased,

whereas the curvature in the plane of the paper is unaltered. Since the decreased curvature is the inward curvature, the total curvature of the displaced film is everywhere outwards, and, therefore, at every point it moves towards its equilibrium position. For this type of displacement the film is, therefore, stable.

If we consider the same type of displacement for the other catenary the first step in the displacement produces a curve which passes above A and B, and, therefore, the second step is to displace the whole curve downwards. This increases the curvature in the plane normal to the plane of the paper (*i.e.*, the inward curvature), leaving, as before, the other curvature unaltered. The total curvature will therefore be everywhere inwards, and every point of the film will move away from the equilibrium position. This film is therefore unstable. The limiting condition of stability for this type of displacement is evidently that A and B should lie on the envelope, for then the curve still passes through A and B after the first step in the displacement, and, therefore, no second displacement is needed in either direction. The minimum possible value for the radius of the end divided by the half length of the film is therefore given by y/x for the envelope—*i.e.*, 1·507879. In this limiting case the radius of the end divided by the radius of the neck is equal to 1·808971.

Other types of displacement have been investigated, but for none of them does the film become unstable below this limit. Experimentally, we have measured the limiting ratio of the diameter of the end to the length, and have found the value 1·510. This is the same as the value calculated above within the limits of our experimental error.

This Paper makes no claim to prove that the films are stable for all kinds of displacement within the limits calculated. Such a proof requires a much greater knowledge of mathematics in the student. The proof of instability outside certain limits is, however, comparatively simple, for if a film is unstable for a single type of displacement it is unstable. The converse, however, is quite untrue, as is illustrated by some of the displacements investigated, particularly the symmetrical displacement in the case of the cylindrical film.

Enough has been said perhaps to correct the very loose presentation of the subject which has been current up to the present time, especially in English text-books.

ABSTRACT.

The authors give a simple method of calculating the equilibrium form of a thin film which is a surface of revolution.

They then consider the stability for certain kinds of displacement of three classes of such films, viz., the sphere, the cylinder and the catenoid. The mathematics used is quite elementary throughout and the treatment is rigorous.

DISCUSSION.

Prof. J. W. NICHOLSON thought the Paper was of considerable interest not only to teachers, as suggested by the author, but to physicists in general. He admired the geometrical construction given and the simple way in which the stability or otherwise of a particular form was determined.

XXVI. *On the Theories of Voigt and of Everett Regarding the Origin of Combination Tones.* *By* W. B. Morton, *M.A.*, *and* Mary Darragh, *M.Sc.*, *Queen's University, Belfast.*

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There has been much discussion on the subject of the additional tones heard when two simple tones are sounded together. This has been concerned with the phenomena to be explained as well as with the theory; the "difference and summation tones" of Helmholtz having been set against the "beat-tones" of Koenig. If mn are the frequencies of the upper and lower primary tones, Helmholtz affirmed the existence of the notes $(m-n)$ $(m+n)$, whereas Koenig heard $(m-hn)$ and $(h+1)\,n-m$, where hn and $(h+1)n$ are the two multiples of n lying on opposite sides of m. Within the compass of an octave Koenig's "lower beat tone" coincides with the difference tone, so that the main subject of dispute concerned the existence of $(m+n)$ on the one hand, and $(2n-m)$ on the other. As a matter of pure observation the question appears to have been settled finally by the recent work of Stumpf * and his school, carried out with unique experimental resources and with the skill derived from long experience in this kind of investigation. The result is to confirm the observations of earlier experimenters, and to establish the existence of both Helmholtz's and Koenig's tones. In addition to these a series of fainter combination-tones can be heard under favourable circumstances when the interval between the primaries is not too large. Inside the interval of a minor third the notes $(2m-n)$ $(3n-2m)$ $(3m-2n)$ $(4n-3m)$ $(4m-3n)$ are heard; between a minor third and a major sixth only the first of these. Between the sixth and the octave only the tones $(m-n)$ $(m+n)$ $(2n-m)$ remain. When the interval exceeds an octave there are only $(m-n)$ $(m+n)$, beyond three octaves only $(m+n)$, and beyond the interval 1 : 12 there are no combination tones at all.

As regards the theory of the phenomena, there has been, and still is, much difference of opinion, and the question is much complicated by the uncertainty as to how far the explanation is to be sought in physical, as distinguished from physiolo-

* "Beiträge zur Akustik u. Musikwissenschaft," Heft V, 1910.

gical or psychological, grounds. The fundamental point at issue between the original theories of Helmholtz and Koenig is the validity of Ohm's law, according to which the ear perceives a note of definite pitch only when a simple harmonic component of corresponding period occurs in the vibratory motion which excites the auditory nerves. When two simple tones are sounded together with amplitudes so small that the principle of superposition holds good, then there are, of course, only these two harmonic components in the air-motion when this is analysed in the Fourier manner, but Helmholtz showed that other components may occur in the motion of the drum of the ear owing to its unsymmetrical structure. His theory relies on the distortion of the aerial motion in the transmitting mechanism of the ear.

Koenig, on the other hand, took a wider view of the capabilities of the ear in recognising periodicity in the incident motion. He supposed that periodic variations of intensity, or "beats," could give the sensation of a tone when sufficiently rapid.

The two theories* mentioned in the title of this Paper may be regarded as extensions of the theories of Koenig and Helmholtz respectively. Voigt goes still further than Koenig in his view regarding the periodicities which the ear can interpret as a tone; Everett seeks to extend the possibilities arising from distortion by the transmitting mechanism.

Both theories receive a good deal of prominence in Auerbach's important work on acoustics (Vol. II. of Winkelmann's "Handbuch der Physik"). An account of Everett's theory is given in Barton's "Text-book of Sound." Some criticisms of Voigt's theory will be found in a Paper by Waetzman,† to which further reference will be made.

The purpose of the present Paper is to point out some objections to Voigt's theory and to give an account of some work which we have done to test the application of the theory of Everett.

Voigt's Theory.

Voigt's theory may be described as a geometrical one. The combination tones are supposed to be determined by certain sine-curves which can be drawn through the extreme points of

* Voigt, "Wied. Ann.," XL., 652, 1890. Everett, "Phil. Mag.," XLI., 199, 1896.

† Waetzmann, "Phys. Zeitschr.," XII., 231, 1911.

the compound vibration-curve representing the motion of the air. It is as if the ear had the faculty of constructing a sine-curve through a number of given points and of recognising the periodicity of the diagram so constructed. Proceeding on this basis, it is shown that curves of the frequencies corresponding to the difference and summation tones can be obtained when the two primary vibrations have the same energy, while the tone $(2n-m)$ is got when the higher primary tone is much weaker than the lower. The special attractiveness of the theory lies in its power to include the Koenig as well as the Helmholtz tones. On this account Auerbach regards it as having finally settled the question, leaving only points of detail for further discussion. His opinion must carry great weight, and it may therefore be worth while to call attention to some difficulties which appear on a close examination of the theory, and which seem to render it untenable as regards the summation tone in particular.

Take, first, the case of equal energies of the primary tones. Their amplitudes are then inversely as their frequencies, and the compound vibration curve has the form

$$y=n \sin mx+m \sin n(x+\delta).$$

It is convenient to take 2π as unit of time and so to call mn the frequencies of the notes.

The stationary points of the curve are given by

$$x=\{(2k+1)\pi+n\delta\}/(m-n),$$

and

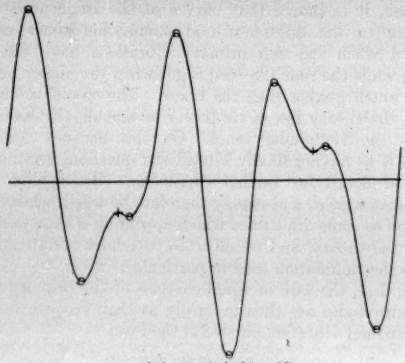
$$x=\{(2k+1)\pi-n\delta\}/(m+n),$$

where k is any integer.

The two sets of points are thus spaced evenly with frequencies $(m-n)$ and $(m+n)$ respectively. We shall call them the $(m-n)$ set and $(m+n)$ set. Each set is now divided into two half-sets by taking alternate points. Through each half of the $(m-n)$ set a sine curve of frequency $\frac{1}{2}(m+n)$ can be drawn, the curves for the two half-sets being opposite in phase. Now, a diagram composed of two equal opposite sine-curves obviously repeats itself in a period half that of the single curves. Therefore, by this construction the points of the $(m-n)$ set are arranged on a diagram which recurs with frequency $(m+n)$. According to Voigt, this fact is perceived by the ear as the summation-tone. In the same way the difference-tone is explained by the similar construction of two sine curves of frequency $\frac{1}{2}(m-n)$ through the two halves of the $(m+n)$ set of

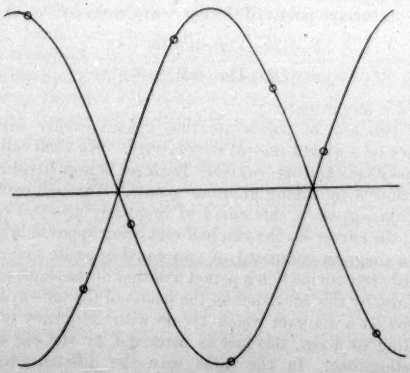
stationary points. The much greater difficulty of hearing the summation-tone is explained by the fact that its curves have to be drawn through a smaller number of points.

Fig. 1 shows the vibration curve for $m=5$, $n=3$, $\delta=6$ deg.



$y=3\sin 5x+5\sin 3(x+6°)$.

FIG. 1.



$y=\pm 5\cos(x-9°)$.

FIG. 2.

The stationary points of the eight-set are marked with circles and those of the two-set with crosses. In Fig. 2 the former set is shown on the sine curves which determine the difference tone 2, and in Fig. 3 the similar construction for the summation tone 8.

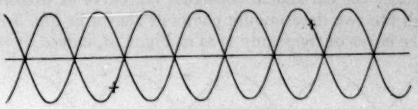
The first objection to be argued against this theory is that a whole series of simple harmonic curves can be drawn through the assigned points in addition to those mentioned above. It is easy to verify that all the points of the $(m+n)$ set lie on all the curves of the family

$$y=(-1)^h(m+n)\sin\{hmx+(h+1)n(x+\delta)\},$$

where h is any integer, positive or negative, while the separate half-sets lie on the pairs of curves given by

$$y=\pm(m+n)\cos\{\tfrac{1}{2}(2h-1)mx+\tfrac{1}{2}(2h+1)n(x+\delta)\}.$$

Taking all the points together, we get in this way the range of



$x=\pm 2\cos(4x+9°)$.

FIG. 3.

frequencies given by the arithmetic progression which has $-m$, n for two consecutive terms (no regard being paid to sign)—*i.e.*,

$$\ldots -(3m+2n),\quad -(2m+n),\quad -m,\quad n,\quad (m+2n),\quad (2m+3n).$$

On the other hand, if we divide the points into half sets and double the frequencies, as Voigt does, we get the progression with $(m-n)$ for central term and common difference $2(m+n)$—viz., $\ldots -(3m+5n), -(m+3n), (m-n), (3m+n), (5m+3n)\ldots$

Waetzmann (*loc. cit.*) has noticed that the complete group of stationary points lies on curves of the fundamental frequencies m, n and amplitude $(m+n)$, and urges as an objection to Voigt's theory that this should introduce the *octaves* of the primary tones. This does not appear to be valid; it is only when the half-sets are grouped on sine-curves with opposite phases that the frequency should be doubled.

It may be urged in defence of the theory that the difference-tone is singled out by the ear because it corresponds to the

sine-curve of smallest frequency, the primary frequencies m n being left out of account. Even this consideration fails in the case of the summation-tone. The points of the $(m—n)$ set are found to lie on

$$y=(-1)^h(m-n)\sin\{hmx-(h-1)n(x+\delta)\},$$

and the half-sets on

$$y=\pm(m-n)\cos\{\tfrac{1}{2}(2h-1)mx-\tfrac{1}{2}(2h+1)n(x+\delta)\},$$

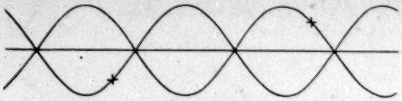
giving the series of frequencies

$$\ldots(3n-2m),\ (2n-m),\ n,\ m,\ (2m-n),\ (3m-2n)\ldots.,$$

and

$$\ldots(5n-3m),\ (3n-m),\ (m+n),\ (3m-n),\ (5m-3n)\ldots.$$

The summation-tone does not, in general, correspond to the lowest frequency in its series. For example, in the case of the interval 3 : 5 we have $3n-m=4$. It is difficult to see any reason why the ear should construct the curve of frequency 8 through the two determining points as shown in Fig. 3 rather than the curve of frequency 4, as in Fig. 4, if, indeed, it is con-



$y=+2\cos(2x+27°)$.

Fig. 4.

ceivable that any curve having a physical significance should be determined by such scanty data.

Another objection to this way of explaining the summation-tone lies in the fact that the $(m—n)$ stationary points supposed to determine it are minima of the elongation from the mean position, whereas the $(m+n)$ points for the difference tone are maxima. We have

$$\frac{d^2y}{dx^2}=-mn\{m\sin mx+n\sin n(x+\delta)\}.$$

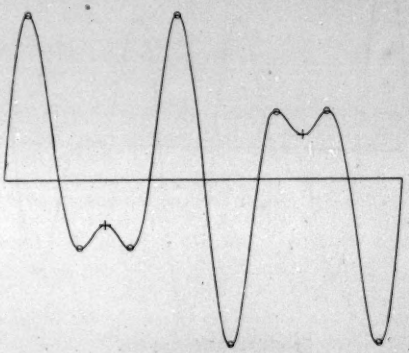
At a point of the $(m-n)$ set $\sin mx=-\sin n(x+\delta)$, which gives $\frac{d^2y}{dx^2}=mny$.

So when y is positive its value is a minimum, and when it is

negative the value is a maximum; in either case we have a point of nearest approach to the axis. For points of the $(m+n)$ set, on the other hand, $\frac{d^2y}{dx^2}=-mny$. It seems scarcely likely that the two classes of points should both give rise to tone-sensations, differing only in intensity.

Again, it appears to be a necessary requirement of a satisfactory theory that it should hold equally well for different phase-relationships of the two primary vibrations. It should also hold through a fairly wide range of relative intensities of the primary sounds. Voigt's theory does not stand these tests.

The effect of varying the phase angle, keeping the energies equal, is shown on Figs. 5, 1 and 6, corresponding to $\delta=0, 6°, 12°$,



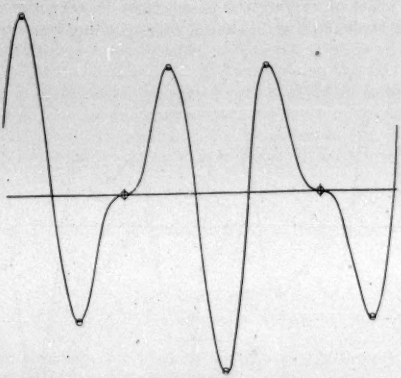
$y=3\sin 5x+5\sin 3x.$

FIG. 5.

for the interval 3 : 5. All the different shapes of the compound curve are obtained as δ increases from 0 to π/mn. It will be seen that the points of the $(m-n)$ set marked as before with a cross, begin by being midway between a pair of points of the other set. In this case their ordinates are most different from those of the adjacent stationary points. With increasing δ the $(m-n)$ points move up to neighbouring points of the other set, fusing with them on the axis for $\delta=\pi/mn$. In this last case, of course, the points are not available for determining any

curve, and in general it is clear that the distinctness of the summation-tone would vary greatly with changes of δ.

When the energies of the primary vibrations are unequal the positions of the stationary points on the curve cannot be expressed by a simple formula, but the progressive modification of the form of the curve can be followed by tracing what becomes of the $(m-n)$ minima, starting from the equal-energy case. If the energy of the upper tone is increased the indentation at each minimum becomes deeper until the axis is reached and crossed, so that the point becomes a maximum on the opposite side. Meanwhile the former maxima persist, so that in the end we get $(m+n)+(m-n)=2m$ peaks on the curve



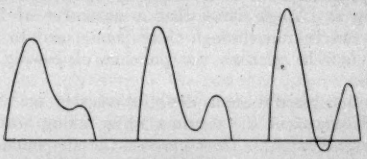
$y=3\sin 5x+5\sin 3(x+12^{\circ})$.

FIG. 6.

corresponding to the curve for the upper note alone. This change is shown on Fig. 7. On the other hand, if the amplitude of the lower note is increased, the depressions at the minima are smoothed out, carrying along with them in each case an adjacent maximum. The combination of minimum and maximum is replaced by a kind of shoulder on the curve which becomes less and less conspicuous, as the upper note is made relatively weaker (Fig. 8). Ultimately the curve has $(m+n)-(m-n)=2n$ peaks, the number for the lower tone. It will be seen that the stationary points supposed to deter-

mine the summation-tone cease to exist when the ratio of the energies is but little different from unity.

The $(m-n)$ minima, and the shoulders which replace them, divide the peaks of the curve into groups, which thus recur with the frequency of the difference-tone. In this grouping we have evidently what remains of the "beats" shown on the



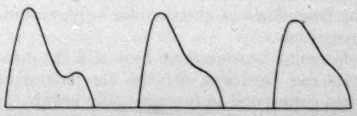
$a=3$ $b=5$ | $a=4$ $b=5$ | $a=5$ $b=5$

$y=a \sin 5x+b \sin 3(x+6°)$.

FIG. 7.

compound curve for two notes of nearly the same pitch, so that we have really come back to Koenig's original explanation of the difference-tones.

It may be mentioned that there is no obvious grouping of the peaks to correspond to Koenig's tone $(2n-m)$, except in cases where $(2n-m)=1$, so that the complete curve is repeated in this period. For example, when the curve for the ratio 5 : 7 is



$a=3$ $b=5$ | $a=2{\cdot}5$ $b=5$ | $a=2$ $b=5$

$y=a \sin 5x+b \sin 3(x+6°)$.

FIG. 8.

drawn the peaks fall into two groups in each complete period, separated by minima, but there is no grouping into three.

Passing now to the other special case considered by Voigt when the upper tone is very weak compared with the lower, it is assumed that the peaks of the compound vibration-curve have the same *positions* as the maxima of the curve of the

lower note alone, the presence of the weak upper tone merely altering the *heights* of the maxima. Thus, the peaks vary above and below their mean level. The deviations from this level are given by the magnitudes of the ordinates of the sine-curve for the upper note, taken at the positions of the maxima of the lower note. In other words, we take a series of equidistant points, spaced with frequency n, on a sine-curve of frequency m. Voigt shows that a sine-curve of frequency $(2n-m)$ can be run through these points, and to this geometrical fact he ascribes the presence of Koenig's second beat-tone.

In the first place it seems doubtful whether one can ignore the displacement of the maxima while taking into account their change of height. If we represent the compound curve by

$$y=c\sin mx+\sin n(x+\delta),$$

where c is small compared with unity, we find that the positions of the maxima are given, to the first order of approximation by

$$n(x+\delta)=(2h+1)\frac{\pi}{2}+\frac{m}{n}c\cos\frac{m}{n}\left\{(2h+1)\frac{\pi}{2}-n\delta\right\},$$

and the maximum ordinate by

$$y=1+c\sin\frac{m}{n}\left\{(2h+1)\frac{\pi}{2}-n\delta\right\},$$

so that the two effects in question are in general of the same order of magnitude.

But if this point be waived we have still the difficulty that other curves can be drawn through the determining points which do not correspond to tones actually heard. In fact, all the curves of the family $y=\sin\{(kn+m)x+\delta\}$ pass through the points where the curve $y=\sin(mx+\delta)$ is met by the ordinates at $x=0, \frac{2\pi}{n}, \frac{4\pi}{n}$, &c., k being any positive or negative integer. Thus, for the interval 3 : 5 we should have the series of frequencies 7, 4, 1, 2, 8, 11.

In view of these difficulties, it seems impossible to regard Voigt's theory as satisfactory, except possibly for the difference-tone, where it may be regarded as a definite form of Koenig's original theory. Further, the trend of recent work is in favour of the validity of Ohm's law. No clear case has been

established in which a tone is given by a mere periodicity, apart from the presence of a simple harmonic component in the analysis of the vibrations.

On the other hand, it should be pointed out that Voigt's theory has since the date of its publication received some experimental support from the fact discovered by Meyer, and confirmed with some modification by Stumpf—namely, that the difference-tone is more easily heard when the lower note is weak and the beat-tone when it is strong. Another supposed point in its favour has been mentioned by Waetzmann, who finds in it an explanation of the fact that difference-tones are not heard for intervals above the octave, when they lie between the pitches of the primary tones. Waetzmann draws the vibration-curve for 1 : 3, and points out that the $(m+n)=4$ maxima, which determine the difference-tone are all of the same height, and would not, therefore, be likely to impress the ear and give rise to a tone. However, this conclusion depends on the fact that the curve is drawn with no phase-difference between the primary vibrations. If the more general case $y=3\sin x+\sin 3(x+\delta)$ is plotted for different values of δ, it is found that the maxima are unequal and that the division of the maxima into groups by the minima or "shoulders" on the curve still occurs.

Everett's Theory.

In Everett's theory stress is laid on the distortion which the vibration-curve of the air-motion must undergo in its passage through the mechanism of the ear. The analysing function of the ear is applied to the vibrations at the inner end of this mechanism, so the result of the analysis will be affected by the distortion. In Everett's opinion the frequency most likely to be introduced is the highest common measure of the primary frequencies; the tone of low pitch of which both primaries are harmonics, having the period in which the complete vibration-curve is repeated. When one analyses a periodic curve of any random form one generally finds the fundamental component coming out strongly. Curves built up of two simple harmonic curves are exceptions to this rule, but if the curve is knocked out of shape in any manner and then analysed the exceptional property of not containing the fundamental would be lost, and this component would generally appear in the analysis.

The experimental work has not confirmed this conclusion of

Everett's. The common fundamental is only heard when it happens to coincide with a difference or beat-tone. Notwithstanding this, the suggestion of Everett seems worthy of further investigation. It is beyond doubt that some distortion takes place, and has an important effect on the quality of the tone heard.

One way of testing the matter is to assume a certain definite type of distortion, and to examine by actual analysis the new frequencies which it introduces into the vibration-curve. In seeking for a reasonable hypothesis to work on one naturally thinks of giving weight to the asymmetry of the transmitting mechanism, in virtue of which it must treat differently the displacements in opposite directions. A simple assumption, and one which lends itself readily to examination in detail, is that displacements in one direction are reduced in a definite ratio as compared with those in the opposite direction. It is not suggested that any definite mechanical arrangement will produce exactly this alteration in the incident vibrations. All the evidence goes to show that the actual procedure in audition is very complicated, depending on physical, physiological and psychical factors. There may, therefore, be some justification for the purely geometrical method here proposed which keeps clear of any special assumption of a mechanical kind. It is perhaps allowable to say that something like this kind of distortion will be caused by a transmission process which gives differential treatment to the inward and outward movements of the ear-drum.

The proposed modification of the vibration-curve may be effected by paring off a certain small fraction from, say, all the positive ordinates. This is the same thing as adding to the given curve another of small amplitude, which is similar to the given curve, with reversed sign, when the latter lies above the axis, and which runs along the axis when the given curve passes below. The new frequencies introduced by the distortion will be obtained by the analysis of this added curve. The actual amplitudes of the new vibrations will, of course, depend on the amount of distortion—*i.e.*, on the ratio in which the positive ordinates are reduced. But their relative amplitudes can obviously be got by applying harmonic analysis to the positive parts of the original vibration-curve, replacing its negative parts by pieces of the axis. We shall arrive at the same result if the negative parts of the curve are taken instead of the positive.

If $x=a_1$, a_2, a_3, &c., are the zeroes of

$$y=a\sin mx+b\sin n(x+\delta),$$

then we require the Fourier expansion of $f(x)$, defined by $f(x)=y$, when x lies in one of the intervals a_1 to a_2, a_3 to a_4, &c., and $f(x)=0$, when x lies in one of the intervals a_2 to a_3, a_4 to a_5, &c.

This process applied to a single vibration gives the manner in which overtones would be produced by the supposed distortion. The expansion of the function

$$f(x)=\sin x. \qquad 0\angle x\angle \pi.$$
$$f(x)=0. \qquad \pi\angle x\angle 2\pi$$

is

$$\frac{1}{\pi}+\tfrac{1}{2}\sin x-\frac{2}{3\pi}\cos 2x-\frac{2}{15\pi}m4x\ldots-\frac{2}{(4r^2-1)\pi}\cos 2rx-,$$

showing the possibility of the introduction of the octave into the auditory effect produced by physically pure tones.

In the case of a compound curve the calculation of the coefficients becomes rather laborious, especially for the larger frequency numbers. A mechanical analyser (Stärzl type) was used in the first instance, and the more important cases were afterwards checked by calculation. The primaries were first taken with equal energies, and the following results were found to hold for all values of the phase-difference δ. The components of the two primary frequencies appear with about half their original amplitudes. The difference and summation tones are both prominent. Next to them comes the octave of the lower primary tone. Neither the second beat-tone nor the common fundamental is specially prominent.

The following table gives the amplitudes of the successive components of the analysis applied to $y=n\sin mx+m\sin nx$—*i.e.*, with $\delta=0$, and equal energies :—

$m:n$	1	2	3	4	5	6	7	8	9	10	11	12	13	14
1 : 1	1·00	0·42	0	0·03	0	0·06	0	0·02	0	...	...	...	...	...
2 : 1	1·09	0·66	0·26	0·09	0·06	0·04	0·03	0·02	0·02	...	...	...	...	...
3 : 2	0·65	1·51	1·13	0·45	0·57	0·14	0	0	0·12	...	...	...	...	...
4 : 3	0·92	0·19	2·00	1·50	0·17	0·56	0·86	0·23	0·04	...	...	...	...	...
5 : 2	0·21	2·50	0·65	0·96	1·00	0·10	0·60	0·13	0	...	...	...	...	...
5 : 3	0	0·82	2·50	0·06	1·50	0·74	0	0·95	0	...	...	...	...	...
5 : 4	1·19	0·23	0·14	2·46	2·01	0·11	0·14	0·56	1·17	...	...	...	...	...
6 : 5	1·51	0·20	0·11	0·06	2·89	2·58	0·09	0·12	0·22	...	1·20	...	...	...
7 : 4	0·17	0·08	1·36	3·50	0·36	0·09	2·05	1·07	0	...	1·09	...	...	...
7 : 5	0	1·44	0	0·28	3·50	0	2·50	0·49	0	0·94	0	1·50	...	...
8 : 5	0·07	0·07	1·53	0·05	3·99	0·21	0·33	2·51	0·19	...	...	...	1·37	...
9 : 5	0·02	0·06	0	1·57	4·50	0·36	0	0·14	2·50	...	...	...	...	2·92

Introduction of a phase-difference made only a slight difference in the relative magnitudes of the components. For example, for the ratio 5 : 3 the successive terms in the analysis of $y=3\sin 5x+5\cos 3x$ are

$$0,\ 0{\cdot}87,\ 2{\cdot}50,\ 0{\cdot}21,\ 1{\cdot}50,\ 0{\cdot}78,\ 0,\ 0{\cdot}91,\ 0,$$

which do not differ much from the numbers given in the table for $3\sin 5x+5\sin 3x$.

It is not difficult to see the mathematical reason for the comparative largeness of the terms of frequencies $(m-n)$ $(m+n)$ and $2n$. The expressions for the terms of frequency r in the analysis of the positive parts of the curve

$$y=a\sin mx+b\sin n(x+\delta) \text{ are}$$

$$\sin rx\left\{\frac{a}{2\pi}\left[\frac{\sin(m-r)x}{m-r}-\frac{\sin(m+r)x}{m+r}\right]\right.$$
$$\left.+\frac{b}{2\pi}\left[\frac{\sin(n-r)x+n\delta}{n-r}-\frac{\sin(n+r)x+n\delta}{n+r}\right]\right\}$$
$$-\cos rx\left\{\frac{a}{2\pi}\left[\frac{\cos(m-r)x}{m-r}+\frac{\cos m+r)x}{m+r}\right]\right.$$
$$\left.+\frac{b}{2\pi}\left[\frac{\cos(n-r)x+n\delta}{n-r}+\frac{\cos(n+r)x+n\delta}{n+r}\right]\right\},$$

where the expressions in square brackets are taken between the limits corresponding to the positive values of y. Thus, the quantities entering are of the form $(-\sin pa_1+\sin pa_2-\ldots)/p$ and the same thing with cosines, where y is positive between $x=a_1$ and $x=a_2$, and so on. When these expressions are evaluated it is found that the greatest numerical value is got for either the sine or the cosine expression when $p=n$.

To see the reason for this it is to be noticed that in the cases considered the number of the roots corresponds to the case of the lower note by itself, the positions of the roots being displaced in consequence of the presence of the upper note. If, now, the lower note were alone we should have $a_2=a_1+\frac{\pi}{n}$, $a_3=a_1+\frac{2\pi}{n}$, &c., so $na_2=na_1+\pi$, $na_3=na_1+2\pi$, &c. Accordingly all the terms would have the same sign and magnitude, and the expressions would reduce to $-2n\sin na$ and $-2n\cos na$ respectively. If the phase-angle δ is such that one of these is small, the other will be large. If the integer p is different from n

some of the terms in the expressions will be positive and others negative, so that the sum is comparatively small. Of course, coincidence of sign is got again when $p=3n$, but the value is reduced by the division by p. When the note m is added the roots are displaced, but when the energies are equal the amplitude of the upper note is less, and so the positions of the roots continue to be fairly evenly spaced along the range 0 to 2π, and the above argument still applies. Therefore, we should expect to find comparatively large Fourier coefficients in cases where one of the numbers $(m-r)$ $(n-r)$ has the value $\pm n$, *i.e.*, when $r=m+n$, $m-n$ or $2n$.

The prominence of these frequencies persists when the energies of the primaries are made unequal. To obtain a measure of the relative loudness of, say, the difference-tone we may divide the square of its amplitude by $(a^2m^2+b^2n^2)$,

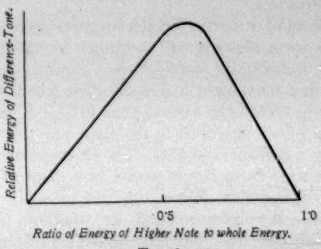


FIG. 9

representing the total energy of the two primaries together. Estimated in this way the maximum value is found when the higher primary tone is somewhat louder than the other. The curve (Fig. 9) shows the variation of the relative loudness of the difference-tone for the interval 3 : 5, the abscissa being the ratio of the energy of the upper primary to the total energy of both primaries. The curve for the summation tone follows a similar course.

It will be seen that the result of this particular kind of distortion affords no support for Everett's contention that the common fundamental would be the most prominent tone introduced. In this it is in agreement with the facts of observation.

On the other hand, no help is given to the theory on the

points which were sources of difficulty in the original explanation of Helmholtz—viz :—

1. The comparative weakness of the summation-tone. Here, as in Helmholtz's theory, the difference and summation-tones appear with amplitudes of the same order of magnitude, so the latter, having greater frequency, should have greater physical energy.

2. The disappearance of the difference-tone with wide intervals. Its relative energy, measured as explained above, rather increases as the interval ratio becomes greater.

3. The presence of the second beat-tone of Koenig. The frequency $(2n-m)$ does not appear in the analysis of the compound curve when the upper note is made relatively weak. There is, of course, the possibility that the explanations of these three points are to be sought in other than purely physical considerations.

Another kind of distortion which suggests itself is that which might arise from slipping at a contact in the transmitting mechanism; for example, between the "hammer" and "anvil" bones. If, in a movement in one direction, there were a small amount of slip before the second bone followed the motion of the first, the effect would be to decrease, say, all the positive ordinates by a definite amount. The representative point on reaching the axis from below would first slide along it for a short distance, and then follow a course parallel to the undistorted curve. We have applied the method of harmonic analysis to this case also, but have found no results of general application.

ABSTRACT.

Voigt connects the existence of difference and summation tones with the fact that the stationary points of the compound vibration-curve, when the primary tones have equal energies, can be grouped in a certain way on sine curves, which recur in the periods of these combination tones. As against this view it is urged (1) that the same points can equally well be grouped on a whole series of curves with other frequencies; (2) that the distinctness of the combination-tones would on this theory vary greatly with phase-relationship of the primaries; (3) that the tones would disappear when the energies of the primaries are not very unequal. Voigt applies a similar method to the case where the upper tone is weak compared to the lower to account for Koenig's second beat-tone. The first of the above objections again applies.

Everett supposed that the distortion of the vibration-curve in passing through the ear would bring in the note whose frequency is the highest common factor of the primary frequencies. The effect

of a special kind of distortion has been tested, consisting in a proportional reduction of ordinates in one direction. The result does not confirm Everett's contention, but shows the appearance of the summation and difference tones.

DISCUSSION.

Mr. F. J. W. WHIPPLE thought the matter was an extremely interesting one, and that the type of distortion assumed by the authors was very feasible. He suggested that an amplification of the theory might be applicable to musical instruments. For example, much of the beauty of the tones of a violin was attributed to the presence of many additional notes besides the fundamental. Quite possibly an asymmetry in the vibrations of the sounding board was responsible for the addition of combination tones of the higher harmonics.

Mr. D. OWEN referred to the case of a man who had no ear-drums, but could nevertheless hear combination tones, which Prof. Morton had cited as an instance in which unsymmetrical vibration of the receiving system could not be the cause of these tones. He presumed that this subject was very deaf, and that any sounds which he could hear at all would be very loud. Consequently, there would be violent displacements of the bones of the ear, and one would expect considerable asymmetry in their vibrations.

Dr. RUSSELL thought that the authors had without doubt completely demolished Voigt's theory, which seemed to have no physical basis whatever. With regard to the unsymmetrical vibrations of the drum, he supposed Helmholtz in his investigation had treated the tympanum as a thin plate, but the attached bones and adjacent fluid would render the problem much more complex.

Prof. S. P. THOMPSON communicated: Though it is well to discuss the theory of combination tones it is much more important to be certain first *what are the facts*, to account for which some theory is required. Hence I raise the vital question of what the facts are: According to von Helmholtz, the combination tones are of two kinds, differential and summational. That is to say, when two tones of respective frequencies m and n are simultaneously sounded, the ear hears the *difference* tone, of frequency $m-n$, and the *summation* tone, of frequency $m+n$; and these are ear-manufactured, and cannot be heard in any resonator, and have nothing to do with beats. According to Koenig, the combination tones are simply beat-tones (and were so regarded also by Thomas Young), and cannot be heard in any resonator. If any supposed combination tone is heard in a resonator, that is a proof that it exists objectively in one or other of the two primary tones, and that, therefore, one or both of the two primary tones is not a *pure* tone but contains the alleged combination tone as a harmonic. Koenig found that beats and beat-tones fall into two series (called superior and inferior), corresponding respectively to the two remainders—positive and negative—to be found by dividing the frequency of the higher tone by that of the lower tone. Thus, let the two primary frequencies be 40 and 74. Then, if we divide 74 by 40, it goes once with a positive remainder of 34, or it goes twice with a negative remainder of 6. Koenig heard, in these circumstances, a rapid beat (inferior) of 34 and a slow beat (superior) of 6. If we take as primaries the note of frequency 3,328 and the note 2,048, there will be heard, according to von Helmholtz, the difference tone 1,280, and the summation tone 5,376. According to Koenig, there will be heard the (inferior) positive remainder 1,280 and the (superior) negative remainder 768. As a matter of fact, two tones are heard, the stronger being 768, and the weaker 1,280. The summation tone 5,376 is not heard at all. If we take, again, two forks giving

2,304 and 1,024, the only combination-tone heard is 256, which is neither difference nor sum. It is the (positive) remainder heard by dividing 2,304 by 1,024. Voigt's theory accounts for this, von Helmholtz's does not. Koenig, who in his life-time tuned with his own hands some tens of thousands of forks, and used the beat-tones to determine the tuning of his high-frequency forks of inaudible pitch, had never been able to hear any tones corresponding to the summational numbers. They simply do not exist if the primary tones are pure.

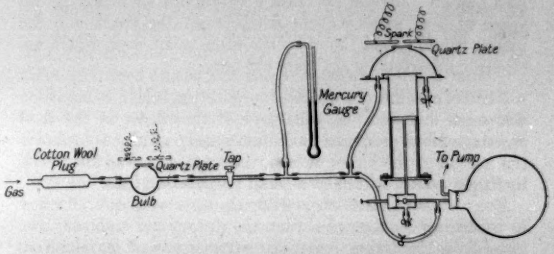
Prof. MORTON, in reply, said he could not say how far combination tones might be produced by the sounding board of a violin. They certainly could be produced by the membrane of a telephone receiver. He had not quoted the man with no ear drums as a case in which asymmetry was not responsible for these tones, but only as one in which we had to go further back than the tympanum for the seat of this asymmetry. A German author had tried to locate this in the motion of the fluids of the ear. Helmholtz's investigation of the vibration of the drum was even less satisfactory than the chairman had supposed, as he treated it simply as an oscillating particle. The analysis had more recently been extended to the case of a membrane and similar results obtained, but even this was still remote from the actual case of the ear drum. With reference to Prof. S. P. Thompson's remarks, the work of observers since the time of Koenig had confirmed his statements as to the presence of the beat-tones; but not his denial of the summation-tones.

XXVII. *Experiments on Condensation Nuclei produced in Gases by Ultra-violet Light.* By Miss MAUD SALTMARSH, *Bedford College.**

RECEIVED MARCH 2, 1915.

THE experiments described in this Paper were performed in the years 1907 and 1914. Those on the action of an electric field were carried out at the Cavendish Laboratory, Cambridge, in 1907, and the others at Bedford College, Regent's Park, in the summer of 1914. During this interval a very exhaustive research on the action of ultra-violet light on gases was made by Lenard and Ramsauer, and published by them in the "Sitzungberichte" of the Heidelberg University, and it was not until after the completion of these observations that their Paper was brought to my notice.

Lenard and Ramsauer used a steam jet to observe condensa-



tion, and this is a less sensitive means of detecting nuclei than the expansion apparatus used throughout these observations, and therefore it was thought that a short account of them might be of value.

The Wilson expansion apparatus used is depicted in the diagram. Its principle is well known, and has been fully described by Mr. C. T. R. Wilson,† so that a further description of it is unnecessary. The cloud chamber, however, was different in the present case. In the experiments on the electrical nature of the nuclei, it consisted of the bottom portion of a cylindrical bell jar, 19·1 cm. in diameter and 5·1 cm. high. The edges of it were ground flat, and to the top one a brass plate was sealed. The other edge rested on a rubber ring, placed on the base of the cloud chamber, and

* Communicated by W. Eccles.

† C. T. R. Wilson, "Phil. Mag.," June, 1904, p. 686.

formed with it an air-tight junction. There was a hole 1·8 cm. in diameter in the brass plate, and this was covered by a quartz plate 0·5 mm. thick, through which the ultra-violet light could pass. An electric field could be put on between this disc and another inside the cloud chamber fixed parallel to it, and insulated from the rest of the apparatus.

The source of ultra-violet light was a spark about 3 mm. long between zinc terminals. An induction coil giving a 6 in. spark was used with three or four accumulators, and a Leyden jar was put in the secondary.

The expansion method of observing the condensation is very sensitive; the presence of only one or two nuclei which are "caught" by any given expansion can be easily seen. A fog contains many more drops (and hence nuclei) than a shower, and the drops hang for some time in the air, while the shower falls quickly. In the electrical observations there was an air space of 6 or more centimetres between the spark and the quartz window, and therefore the ultra-violet rays, which are strongly absorbed by air, were cut out.

§ 1. To observe the effect of an electric field a comparison was made between the behaviour of the nuclei in the field and that of ions produced by radium; and, in order to facilitate the motion of the nuclei or ions, the apparatus was filled with hydrogen which contained a small proportion of air.

Even without an electric field, the ions were less effective in producing condensation than the nuclei, and a shower was formed on the latter, when the expansion was too small to "catch" any of the ions. The least expansion required for condensation on negative ions is about 155 mm. of mercury.

There was no marked increase in the density of the cloud on the nuclei, as the expansions were successively increased through this value, which might have been the case had there been many ions present.

The differing effects of the field, which was 50 volts per centimetre on ions and on the nuclei, was very marked with an expansion of 189·5 mm., as shown by the following table:—

Difference of pressure on expansion.	Nature of cloud.					
	No light or radium.		Radium.		Ultra-violet light.	
	P.D. = 0.	P.D. = 50 volts per cm.	P.D. = 0.	P.D. = 50 volts per cm.	P.D. = 0.	P.D. = 50 volts per cm.
189·5 mm.	Large shower	Very small shower	Fog	Large shower	Very large shower	Very large shower

Observations of a similar kind were made, and in none of them were the nuclei appreciably moved by the field.

That the ultra-violet nuclei do not disappear more quickly in the presence of an electric field than without one was demonstrated as follows :—

The time of exposure to the light was arranged so that a very dense shower was produced on the nuclei with an expansion of 172 mm. This shower was the same in the presence or absence of the field. Making the expansion 2·5 minutes after stopping the spark a shower was obtained, and after four minutes a smaller one, but in both case the number of drops was the same, whether the field was on or whether it was not.

With the same expansion and without an electric field a shower of about the same density was produced on ions formed by exposure to radium of suitable duration. If there was no field, and the apparatus was left for four minutes after the radium had been removed, a shower was obtained on expansion of about the same density as that on the nuclei after four minutes ; but in the presence of a field the number of drops formed on the ions was very small after $\frac{6}{7}$ second.

The rates of disappearance of the nuclei and ions seemed to be about the same when there was no electric field ; but, while it had no effect on the rate of disappearance of the nuclei, it increased that of the ions very much.

Lenard and Ramsauer* have divided the action of ultra-violet light on gases into three separate divisions :—

1. The production of electrified particles, sometimes of molecular dimensions and sometimes larger, the latter being formed by ions of molecular size joining on to uncharged nuclei.

2. Chemical action, such as the formation of ozone.

3. Production of nuclei, which serve as centres of condensation.

They showed by electrometer measurements that ultra-violet light, which is highly absorbed by air, is very strongly photo-electric in air, and the permanent gases of it, also in CO_2, NH_3, CS_2, &c.

Ultra-violet light of longer wave-length which passes through several centimetres of air is not photo-electric in the permanent gases of air if they are pure, but is so in atmospheric air, since this always contains other minor constituents, by which the light is absorbed (CO_2, NH_3, &c.). They also main-

* Lenard and Ramsauer, Heidelberg "Berichte," 1910 and 1911.

tain that the nuclei produced in gases by ultra-violet light are much more effective for condensation than gaseous ions.

Accordingly, if both ions and nuclei be formed by the ultra-violet light, the nuclei would be "caught" rather than the ions, and the density of the cloud might not be much affected by the removal of the ions. This would account for the cloud on the nuclei being as dense in the presence of a field as without one.

If, however, some ions were formed by the light, and they joined on to the nuclei to form the larger carriers observed by Lenard and Ramsauer, they should disappear more rapidly when acted on by an electric field than when they merely diffused away. This was not found to be the case, but possibly not many ions were formed by the light.

§ 2. In the following series of experiments another expansion apparatus was used, but it was essentially the same as before, the only difference being in the cloud chamber. This was a 5 in. dialyser, the large edge of which was ground so as to form with vaseline or rubber lubricant an air-tight junction with the brass plate forming the base of the cloud chamber. The small neck of the dialyser was cut off short, and a quartz plate 0·1 cm. thick was sealed to it.

Sometimes the gas experimented upon was collected in a small glass bulb, shown in the diagram. It was about 5 cm. in diameter and had a quartz window, so that the gas could be exposed in it to the action of the light, and could then be drawn into the cloud chamber.

Observations were made with the vapours of other liquids besides water in order to determine if the nuclei were equally effective in producing condensation of them. The water in the expansion apparatus was in each case replaced by the liquid to be experimented on.

The values in the first table give the least expansion required for condensation in dust-free air. This occurs on the negative ions, which seem always to be present in small numbers in atmospheric air.*

Water	$v_2/v_1=1{\cdot}25$
Absolute alcohol	$v_2/v_1=1{\cdot}16$
Toluol	$v_2/v_1=1{\cdot}29$
Turpentine	$v_2/v_1=1{\cdot}30$

* C. T. R. Wilson, "Phil. Mag.," June, 1904.

In observing condensation on the nuclei, in order that they might be exactly the same in all cases and not dependent on the nature of the vapour, dust-free atmospheric air was exposed in the bulb to ultra-violet light from the spark. Since the bulb had not before been in communication with the liquids, it contained none of their vapours, and the nuclei were therefore identical in all cases. They were swept from the bulb into the cloud chamber, in which the air was at diminished pressure, by causing a stream of dust-free air to be drawn through the bulb into the cloud chamber, until the requisite pressure was obtained. The spark was placed near the quartz window of the bulb about 0·5 cm. from it; the light was, therefore, fairly strong and the supply of nuclei plentiful.

From the table it will be seen that the nuclei were effective in producing condensation of all these vapours, and require only small expansions for condensation to occur. Almost the smallest expansion possible to give was sufficient to produce some drops.

Vapour.	Time of exposure.	Difference of pressure on expansion.	Cloud.
Abs. alcohol	15 secs.	2·9 cm.	Shower
Toluol	1 min.	3·1 „	Thin fog
Turpentine	1 min.	5·3 „	Thin fog

Since the nuclei are equally effective in causing condensation of the vapours of water, alcohol, toluol and turpentine, it does not seem likely that the substance of which they consist should, by dissolving in a minute drop of the liquid, lower the vapour pressure, and hence help the drop to grow. The substance would have to be equally soluble in all four liquids. This explanation of the nuclei has been put forward, and a particular case of it is dealt with later on in the Paper.

The effect of exposing air saturated with the various vapours in turn to the direct action of ultra-violet light was then tried. It is well known that if air saturated with water vapour is exposed to the action of very strong ultra-violet light, a dense fog appears in it after a time without any expansion having been made. The ultra-violet light must be very strong to produce this effect. In the present case the effect was not obtained with water vapour even when the spark was placed

as close as possible to the quartz plate, but it was with alcohol vapour.

Spontaneous condensation of alcohol vapour seems to occur on very small nuclei, if one can judge of the size of them by the strength of the light which forms them; but even water vapour would condense with a small expansion on nuclei formed by light which had traversed 50 cm. of atmospheric air between the spark and the quartz plate.

The following table gives some of the observations made with direct exposure of the air in the cloud vessel :—

Vapour.	Time of exposure.	Difference of pressure on exposure.	Distance of spark.	Cloud.
Water	15 secs.	1·7 cm.	0·7 cm.	Thin fog
Water	15 secs.	1·8 cm.	50 cm.	Small fog
Abs. alcohol ...	1 min.	No expansion	0·7 cm.	Fog
Abs. alcohol ...	1 sec.	5·7 cm.	50 cm.	Small shower
Toluol	15 secs.	8·45 „	0·7 „	Large shower
Turpentine ...	1 min.	3·5 „	0·7 „	Thin fog

§ 3. The action of the light to form nuclei in various gases was then observed, condensation of water vapour on them being used to detect the nuclei. In order to determine if the presence of water vapour was necessary for the production of nuclei, dust-free air was passed over H_2SO_4 on pumice stone into the bulb, and left there in contact with P_2O_5 for some time. It was then exposed to the action of the light, and the nuclei were swept into the cloud chambers as before.

Time of contact with P_2O_5	Difference of pressure on expansion	Cloud.
Three hours	8·1 cm.	Very large shower
Two days	8·05 „	Fairly large shower
Air not dried	11·85 „	Very large shower

It is not probable that the very last traces of moisture were removed, but the formation of nuclei did not seem to be dependent on the presence of more than a very minute quantity of water vapour.

Carbon dioxide was then tried, and in this case the cloud vessel was exhausted and filled several times over with CO_2 from a cylinder of liquid CO_2, and the gas was exposed in it to

the light. With an exposure of one minute duration and a difference of pressure of 12·7 cm. on expansion, a dense fog was obtained exactly similar to one obtained in air under the same conditions.

CO_2 and air, probably the oxygen of it, therefore, are equally susceptible to the action of ultra-violet light to produce nuclei.

The next three gases experimented upon showed little or no nuclei formation. First of all, the bulb was filled with impure hydrogen from a Kipp's apparatus, then exhausted to less than 2 cm. pressure by a water pump and re-filled with hydrogen formed by the electrolysis of water. The gas was not dried, and hence water vapour was mixed with it. It was exposed for two minutes in the bulb and then drawn into the cloud chamber and there expanded to a difference of pressure of 11·95 cm. Only a small shower resulted, while with dust-free air treated in the same way a thin fog was obtained. Water vapour alone was then tried. A little distilled water was put into the bulb, which was then exhausted by a very efficient water pump. The remaining water vapour was exposed to light for two minutes, and any nuclei formed were swept into the cloud chamber by passing a current of dust-free air through the bulb. The pressure of the water vapour was 1·55 cm. On expansion with a difference of pressure of 8·05 cm. only a small shower resulted, showing that few nuclei had been formed.

The last gas to be tried was nitrogen, which was obtained from a cylinder of nitrogen. It was tested first by sparking over caustic potash in an eudiometer and found to be very pure. Nevertheless, there was sufficient oxygen for nuclei to be produced in profusion by the ultra-violet light. The cloud chamber itself was filled with the gas, and hence it was saturated with water vapour. In order to rid it of the remaining traces of oxygen it was sparked inside the cloud chamber for 20 minutes between platinum points. Four small vessels of caustic potash stood on the base of the cloud chamber, and the cylinder in which the piston worked contained pure distilled water. Moist filter papers were placed on the metal parts of the cloud chamber to keep the space well saturated.

The table shows the result of exposing moist nitrogen, freed from any trace of oxygen, to the ultra-violet light. From it we see that no nuclei were produced, but when some air was

let in, everything else remaining the same, an exposure of five seconds produced many nuclei.

Gas.	Time of exposure.	Difference of pressure on expansion.	Cloud.
N before sparking ...	1 min.	6·95 cm.	Thin fog
N after „ ...	1 „	6·75 „	No drops
„ „ „ ...	7 „	9·25 „	No drops
„ „ „ ...	13 „	11·55 „	One or two drops
„ „ „ ...	19 „	15·2 „	Small showers
Air let in	5 secs.	9·65 „	Small fog

In all the observations with various gases the distance of the spark from the quartz window was 0·7 cm. or less.

Lenard and Ramsauer* have shown that by removing all traces of such substances as H_2O, NH_3, CO_2 from air or oxygen no nuclei which would affect a steam jet were formed by ultra-violet light. They entirely removed these substances by passing the gas through a vessel surrounded with liquid air, and thus condensing them out. The walls of the connecting tubes and the vessel in which the gas was exposed to the light had also to be very scrupulously cleansed by heating before the gas could be entirely freed of these impurities. Unless so cleaned, the walls of the vessel were able to give off a sufficient quantity of these substances for the production of nuclei, although the gas itself had first been purified by cooling; but they found that unless oxygen or CO_2 were also present no nuclei were formed.

The results of the experiments which have just been described agree with those of Lenard and Ramsauer in showing that, unless oxygen or CO_2 are present, ultra-violet light produces no nuclei. The small traces of necessary impurities must in each case have been present, but they alone are not sufficient for the production of nuclei.

§ 4. It has been suggested that ultra-violet light acting on moist air might result in the formation of particles of H_2O_2, which, by dissolving in the small drops of water, would help them to grow larger and become stable.†

An attempt was made to test this point directly by a chemical method. The quantity of H_2O_2 formed would certainly be very small, but there are very delicate tests for it; and a preliminary experiment made with actual particles of H_2O_2

* Lenard and Ramsauer, Heidelberg "Berichte," 1910 and 1911.

† J. J. Thomson, "Conduction of Electricity Through Gases," p. 140.

showed that it was possible to detect their presence in a cloud. Titanic acid, which is ordinarily colourless, becomes yellow when a very small trace of H_2O_2 is added to it, and this was used as a detector.

First of all the cloud chamber was set up with water and dust-free air inside, so that unless the expansion was large enough to catch ions, no drops were formed. Then a small quantity of solution of H_2O_2 was let into the cloud vessel through one of the tubes in the base of it. No ultra-violet light was used, but a few drops were always obtained on expansion. The number of drops depended on the time between successive expansion. When they were made as fast as possible one after the other a few drops were formed just over the solution. If time were allowed, the particles diffused to other parts of the vessel. These nuclei must have been particles of H_2O_2, perhaps already dissolved in minute drops of water; but they cannot have been larger than the ultra-violet nuclei, for in all cases an expansion was required to form condensation on them.

The table gives some of the observations :—

Time for diffusion.	Difference of pressure.	Cloud.
A few seconds	3·5 cm.	Few drops just above solution
5 minutes	3·5 „	Very small shower near solution
30 minutes	3·1 „	Small shower in whole vessel

The presence of H_2O_2 in these showers was easily detected with titanic acid.

The apparatus was cleaned and the base of the cloud vessel covered with filter paper well soaked in solution of H_2O_2. It was then set up again with water and air, and a small open glass dish containing 0·44 gramme of titanic acid was placed in the cloud vessel in such a position that part of the clouds would fall into it.

Expansions were made with a difference of pressure of about 9·35 cm., and each time a small shower was produced, part of which fell into the acid. After 19 expansions, taking altogether half an hour, the acid appeared slightly but distinctly yellow, owing to particles of H_2O_2 having fallen into it with the showers.

The apparatus was then thoroughly cleared of H_2O_2 and

filled with dust-free air and water, about the same quantity of titanic acid having been placed inside the cloud vessel. The air was exposed to light from the spark, which was 0·7 cm. from the quartz plate.

One hundred and fifty expansions were made of 11·5 cm. difference of pressure, and the clouds produced were very much larger than the small showers on the H_2O_2 particles. They were allowed to fall into the acid, but no colouring of it could be observed, although the nuclei were as effective in producing condensation as the H_2O_2 particles—that is, a cloud could be formed on them with quite as small an expansion as on the H_2O_2 particles.

The amount of H_2O_2 formed by the action of the light must therefore, at the most, be only a small fraction of the whole change which occurs in nuclei formation.

§ 5. That ozone is produced by the action of ultra-violet light on air and oxygen has been demonstrated by many observers (recently by Pring, Proc. Roy. Soc., May 1, 1914). Lenard and Ramsauer* obtained evidence of ozone in oxygen which had been exposed to the action of light, and which beforehand had been freed from impurities by cooling; but they found that there were no condensation nuclei. If the small traces of impurities were not removed nuclei were formed in abundance. Ozone, together with these impurities, might, therefore, be one of the means by which the nuclei are formed. Experiments bearing on this point were carried out as follows :—

Oxygen from a cylinder was dried by bubbling through H_2SO_4, and was passed into a silent discharge ozoniser, and from it into the cloud chamber, which had been partially exhausted of air. On entering the cloud chamber condensation occurred at once in the ozonised oxygen, unless the amount of ozone was very small, in which case an expansion was required. The quantity of ozone was roughly estimated by the time of the discharge, and the table gives the nuclei formation resulting from the production of ozone.

Time of discharge.	Difference of pressure.	Cloud.
1–1½ minutes	No expansion	Extremely dense fog
5 seconds	,, ,,	Fog
½ seconds	,, ,,	Slight condensation
One make and break of coil causing discharge	9·35 cm.	Fog
Smaller quantity still	2·75 cm.	Small shower

* Lenard and Ramsauer, Heidelberg "Berichte," 1910 and 1911.

In the first observation a strip of potassium iodide starch paper placed in the cloud vessel became dark blue; in the second observation a similar strip became slightly coloured; while in the other observations, and about five similar ones, a strip of paper which was present all the time was not coloured at the end, although dense fogs were produced on expansion.

The formation of ozone in oxygen by a silent discharge results also in the formation of condensation nuclei. These nuclei were not only active in causing condensation of water vapour, but also of absolute alcohol, toluol and turpentine vapours. This was demonstrated by observations similar to the above ones, except that the various liquids were in turn put in the expansion apparatus.

§ 6. Experiments were then made to determine whether the nuclei would be destroyed by heating the air containing them. Two quartz tubes about 15 cm. long and 1 cm. in diameter were joined in series between the bulb and the cloud chamber, and each tube was heated for some hours by a Bunsen flame over a length of about 4 cm. in the centre. Dust-free air which had been exposed to ultra-violet light while it was in the bulb, was drawn slowly through the tubes into the cloud chamber and there expanded. A comparison was made with ozonised oxygen, and in both case the nuclei were destroyed.

From the table it is seen that dust-free air drawn slowly through the hot tubes contains practically no nuclei. Also a stream of dust-free air, exposed to ultra-violet light, while it passed through the bulb, contained nuclei sufficient to form a fog if it passed through the cold tubes, but scarcely any if it passed through the hot tubes. Similar results were obtained with the nuclei in ozonised oxygen.

Gas.	Tubes.	Time of discharge or light.	Difference of pressure.	Cloud.
Air	Hot	None	11·6 cm.	Very small shower
Air	Hot	4¼ mins.	11·5 „	Small shower
Air	Cold	4¼ mins.	11·6 „	Fog
Oxygen and ozone	Hot	⅖ sec.	10·25 „	Very small shower
Oxygen and ozone	Cold	⅖ sec.	10·3 „	Fog

In making these observations a curious phenomenon was observed. The result shown in the first line of the table was obtained when the air was drawn slowly through the hot tubes. But if the air, which was originally dust free and was not exposed to ultra-violet light from the spark, was drawn faster

through the hot tubes it was found to contain a very large number of nuclei, so that a fog was produced on expansion. This effect could not be got rid of, although the tubes were heated for a long time and a stream of air was drawn many times through them. It always happened if the air was drawn through the tubes too quickly. Also, there was a comparatively sharp division between the rate of flow of air through the hot tubes when the nuclei were produced in them and the rate when those existing beforehand were destroyed in them.

It is difficult to account for these nuclei, but their existence may possibly be explained thus: Since the tubes were of quartz, the nuclei may have been produced in them by light from the Bunsen flame, which was playing round the outside of the tubes; and these nuclei may have passed away from the hot part of the tubes before they became hot enough to be destroyed. If the passage of the air through the tubes was slower, they and any other nuclei which were there would have time to be heated and destroyed.

One other property of the nuclei which was observed was that it was not possible to draw them with a stream of air through a plug of cotton wool. They were all filtered out of the air like dust particles.

Summary.

1. Nuclei produced in air by ultra-violet light which has traversed a few centimetres of air are not affected by an electric field of 50 volts per centimetre.

2. The nuclei are equally effective in producing condensation of water, toluol and turpentine vapours, and they are formed even by light which has traversed 50 cm. of air.

3. Alcohol vapour condenses without expansion on much smaller nuclei than does water vapour.

4. No nuclei were formed by the light unless oxygen or CO_2 were present in the gas.

5. No trace of H_2O_2 could be detected in the clouds formed on the nuclei.

6. Oxygen containing ozone also contains nuclei for condensation, and these nuclei have similar properties to those formed by ultra-violet light.

7. The nuclei can be destroyed by heating the air containing them.

In conclusion, it seems probable that the nuclei formed by ultra-violet light do not cause condensation by virtue of any

particular chemical composition, but that they are particles large enough to act like dust particles as centres round which condensation can begin. It is probable that some of the dust particles in the air are really nuclei formed by the action of ultra-violet light.

I should like here to express my thanks to Mr. Wilson, Dr. Womack and Dr. J. F. Spencer for the help they have so kindly given me.

ABSTRACT.

1. Nuclei produced in air by ultra-violet light which has traversed a few centimetres of air are not affected by an electric field of 50 volts per centimetre.

2. The nuclei are equally effective in producing condensation of water, toluol and turpentine vapours, and they are formed even by light which has traversed 50 cm. of air.

3. Alcohol vapour condenses without expansion on much smaller nuclei than does water vapour.

4. No nuclei were formed by the light unless oxygen or CO_2 was present in the gas.

5. No trace of H_2O_2 could be detected in the clouds formed on the nuclei.

6. Oxygen containing ozone also contains nuclei for condensation, and these nuclei have similar properties to those formed by ultra-violet light.

7. The nuclei can be destroyed by heating the air containing them.

It seems probable that the nuclei formed by ultra-violet light do not cause condensation by virtue of any particular chemical composition, but that they are particles large enough to act like dust particles as centres round which condensation can begin.

DISCUSSION.

Prof. O. W. Richardson said the author had succeeded in eliminating a number of substances to which these effects had from time to time been attributed. The method employed was extremely sensitive, one or two ions being easily detected. Of the various substances eliminated the only one which he was not quite certain of was ozone. The elimination of this seemed to rest on the results of Lenard and Ramsauer. Did the author think these results were absolutely conclusive ?

Mr. D. Owen said that the explanation of the action of ozone in causing cloud formation was difficult if regarded as a direct effect. Possibly ozone acting on water vapour might give rise to hydrogen peroxide, in which case the suggestion of Mr. C. T. R. Wilson of a lowering of the saturation vapour pressure would apply. However the failure, in the case of fogs produced by ultra-violet light, to detect hydrogen peroxide by the titanium oxide test, which, according to Dr. Senter, can detect one part of hydrogen peroxide in 10 millions, seemed to exclude the action of hydrogen peroxide : this test is, moreover, a specific test for hydrogen peroxide even in the presence of other oxidising agents. As a possible alternative, the agency of oxides of nitrogen might be suggested. It would be of interest to test whether the phenomenon occurs when nitrogen is entirely excluded. In conjunction with chemical tests of amount present, quantitative data enabling the resulting lowering of the vapour pressure to be calculated should enable the problem in question to be definitely solved.

Dr. H. Borns remarked that the detection or identification of hydrogen peroxide was not always easy. In experiments of the kind described in the Paper effects might be ascribed to hydrogen peroxide which were really due to nitrogen oxides. Such oxides would be produced when air was drawn through hot quartz tubes, and might help to account for the peculiar observations mentioned in the last part of the Paper.

Miss Saltmarsh, in reply, was not aware of any work having been done on the effect of oxides of nitrogen in producing condensation nuclei. In reply to Prof. Richardson, Lenard and Ramsauer had not stated that ozone was ineffective. They laid great stress on the presence of minute quantities of impurities. They attempted to get rid of these by condensing them out at low temperatures. They held that the ozone reacting on these traces of impurities was instrumental in producing the effects.

XXVIII. *On the Self-induction of Solenoids of Appreciable Winding Depths.* *By* S. BUTTERWORTH, *M.Sc., Lecturer in Physics, School of Technology, Manchester.*

RECEIVED FEBRUARY 27, 1915.

1.

THE only formulæ which appear to have been given for the self-induction of solenoids in which the correction for winding depth is included are those of Rosa* and Cohen.† In arriving at these formulæ certain doubtful approximations have been made. In the present Paper formulæ have been developed which are free from such approximations,‡ and it is shown that, while Rosa's formula gives better results than that of Cohen, it does not possess the accuracy claimed for it by its author.

2. *Cohen's Method.*

The coil under consideration is divided into a finite number of layers, each of which is sufficiently thin to be treated as a cylindrical current sheet. The self-induction of the coil is then found by combining the self and mutual inductions of these layers according to the usual laws. For the self-inductions Cohen uses the formula,

$$L=4\pi^2n^2a^3\left(\frac{2+c^2}{\sqrt{4+c^2}}-\frac{8}{3\pi}\right), \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

in which a is the mean radius of a layer, ca the coil length, and n the number of turns per unit length.

The formula is an approximation to an exact elliptic integral formula due to Lorenz.

For the mutual inductions he uses

$$M=4\pi^2n^2a^3(\sqrt{c^2+a^2/A^2}-A/a+a/8A), \quad . \quad . \quad . \quad (2)$$

in which a and A are the radii of the inner and outer cylinder

* Rosa, "Bulletin" of the Bureau of Standards, 4, p. 369, 1908.

† Cohen, "Bulletin" of the Bureau of Standards, 4, p. 384, 1908.

‡ The formulæ of this Paper are approximate in that they neglect (*a*) the insulation space between the wires of the coil, (*b*) the effect of the helicity of the winding.

respectively, and ca is the length of the cylinders. This is an approximation to a formula due to Maxwell.*

The final formula obtained by Cohen for the thick coil is

$$L/\pi^2N^2R^3=\frac{4}{m}\left(\frac{2+c^2}{\sqrt{4+c^2}}-\frac{8}{3\pi}\right)$$
$$+\frac{8}{m^2}\left[\left\{(m-1)k_1^2+(m-2)k_2^2+\ldots\right\}\left(\sqrt{k_1^2+c^2}-\frac{7}{8}k_1\right)\right.$$
$$-\frac{\delta k}{2}\left\{m(m-1)k_1^2+(m-1)(m-2)k_2^2+\ldots\right\}\left(1-\frac{k_1}{\sqrt{k_1^2+c^2}}\right)$$
$$\left.-\frac{\delta k}{16}\left\{m(m-1)k_1^2+(m-2)(m-3)k_2^2+\ldots\right\}\right] \quad . \quad . \quad (3)$$

in which m is the number of layers, N the number of turns per unit length (including all the layers), cR the coil length, k_1R, k_2R, . . . the mean radii of the layers ($k_{r+1}>k_r$), and Rδk the width of each layer.

He states that this formula has a minimum accuracy of one-half of 1 per cent. when $c>4$.

An *a priori* test of this assertion may be obtained by an examination of formulæ (1) and (2). Since the distance apart of contiguous layers is small compared with their radii, (2) should approximately agree with (1) when $A=a$, *i.e.*,

$$L=4\pi^2N^2a^3\left(\sqrt{1+c^2}-\frac{7}{8}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

should not differ by more than one-half of 1 per cent. from (1) when $c>4$.

Further, we may test the absolute accuracy of (1) and (2) by means of some exact formula, such as that of Havelock,† viz.,

$$L=4\pi^2N^2a^3c\left\{1-\frac{8}{3\pi c}+\frac{1}{2c^2}\right)-\frac{1\cdot3}{2\cdot3}\frac{1}{c^2}\right)-\frac{3\cdot5}{3\cdot4}\frac{1}{c^2}\right)\cdots\Big\} \quad . \quad (5)$$

in which the notation $)x$ means that any term is obtained from the preceding term by multiplying by x.‡

* Maxwell, "Electricity and Magnetism" (Vol. II., Art. 678).

† Havelock, "Phil. Mag.," Vol. XV., p. 332, 1908.

‡ By using this notation, computation from the formula is far more rapid than when each term in the series is calculated separately.

Table I. gives the values of $L/4\pi^2N^2a^3$ as obtained from the three formulæ.

TABLE I.

c.	Havelock.	Formula (1).	Formula (4).
4	3·2725430	3·1761	3·2481
5	4·2492672	4·1650	4·2240
6	5·2333879	5·1596	5·2078

The values from Havelock's formula are calculated to eight figures as they are needed later. The table shows that neither of the fundamental formulæ used by Cohen are sufficiently accurate to give results to one-half of 1 per cent.

3. *Rosa's Method.*

The inductance (L_0) of a cylindrical current sheet of the mean radius of the thick coil is calculated by any suitable exact formula, and the correction for thickness is got by deducting from L_0 an amount ΔL, where

$$\frac{\Delta L}{\pi^2N^2R^3}=\frac{8}{\pi}cT(A+B), \quad \ldots\ldots\ldots \quad (6)$$

in which N, R, c have the same meanings as before, 2TR is the winding depth, and A and B are tabulated functions of T and c/T respectively. The following values of A and B taken from Rosa's tables are required later :—

TABLE II.

c.	T.	A.	B.
4	1/10	0·6922	0·3099
5	1/12	0·6926	0·3218
6	1/10	0·6922	0·3218

The method of obtaining A and B is to divide the coil into slices by planes perpendicular to the axis, such that each slice forms a coil with a winding channel of square section. The correction A arises from the difference between the self-induction of a coil of square section and that of a current sheet of the mean radius of the coil. Weinstein's formula is used for the square-sectioned coil and Rayleigh's formula for the current sheet. Apart from the question of the validity of Weinstein's formula up to 2T=0·25 (the maximum thickness for which Rosa tabulates A), there appears to be no serious error in this correction.

The correction B is due to the mutual induction between the slices. It is obtained by the method of geometrical mean

distance, and herein lies the most probable source of error, as the method of geometrical mean distance is only valid for coils whose sectional dimensions *and distance apart* are small compared with their mean radii. The latter condition is far from being the case with coils of the nature under consideration. It will be shown in Section 10 that Rosa's correction is 2 per cent. in error for coils whose length is four times their radius, while the error in the total induction is 0·2 per cent.

4. *Present Method.*

The coil is first taken to be a portion of an infinite coil, and the self-induction (L_1) calculated under these conditions. The work to be done to remove the portion from the remainder of the infinite coil is then determined, and this gives the diminution in self-induction due to the effect of the pole faces of the coil. This work is due to the attraction of two pairs of unlike poles in contact, together with the repulsion of two pairs of like poles at a distance apart equal to the length of the coil, or, since the work to be done to separate a pair of poles is equivalent to the mutual induction between the corresponding semi-infinite coils, the end correction to be deducted from L_1 is

$$2M(o)-2M(l),$$

where $M(l)$ represents the mutual induction between two semi-infinite coils at a distance, l.

Hence, if ΔL_1 represents the (additive) thickness correction for L_1, $\Delta M(l)$ that for $M(l)$, the total thickness correction is

$$\Delta L=\Delta L_1-2\Delta M(o)+2\Delta M(l) \quad . \quad . \quad . \quad . \quad . \quad (7)$$

5. *The Correction* ΔL_1.

The self-induction of a length, l, of an infinite coil of outer and inner radii a and b is given by Maxwell* as

$$L_1=\frac{2}{3}\pi^2N^2l(a^2+2ab+b^2),$$

where N is the number of turns per unit length. Hence, if $a=R(1+T)$, $b=R(1-T)$, $l=Rc$

$$L_1=4\pi^2N^2R^3c\left(1-\frac{2}{3}T+\frac{1}{3}T^2\right),$$

so that the thickness correction ΔL_1 is given by

$$\frac{\Delta L_1}{\pi^2N^2R^3}=-\frac{8}{3}cT\left(1-\frac{1}{2}T\right) \quad . \quad . \quad . \quad . \quad . \quad (8)$$

* Maxwell, "Electricity and Magnetism," Vol. II., Art. 679.

6. *The Correction* $\Delta M(o)$.

The mutual induction between two semi-infinite, coaxial, cylindrical current sheets with coplanar ends is

$$m=\frac{8}{3}\pi a\{(a^2+b^2)E-(a^2-b^2)K\} \quad . \quad . \quad . \quad . \quad (9)$$

where a and b are the radii of the two sheets ($a>b$) and K and E are complete elliptic integrals of the first and second kinds to modulus b/a. When b/a is nearly unity, K and E are suitably expressed by the series

$$\left.\begin{aligned} K=&\log\frac{4}{k}+\frac{1^2}{2^2}k^2\left(\log\frac{4}{k}-\frac{2}{1\cdot2}\right)\\ &+\frac{1^2\cdot3^2}{2^2\cdot4^2}k^4\left(\log\frac{4}{k}-\frac{2}{1\cdot2}-\frac{2}{3\cdot4}\right)\cdot\cdot\cdot\cdot\cdot\\ &+\cdot\cdot\cdot\\ E=&\ 1+\frac{1}{2}k^2\left(\log\frac{4}{k}-\frac{1}{1\cdot2}\right)\\ &+\frac{1^2}{2^2}\cdot\frac{3}{4}k^4\left(\log\frac{4}{k}-\frac{2}{1\cdot2}-\frac{1}{3\cdot4}\right)\cdot\cdot\cdot\cdot\cdot\\ &+\cdot\cdot\cdot \end{aligned}\right\} \quad . \quad . \quad (10)$$

in which $k^2=1-\dfrac{b^2}{a^2}$.

If we put $a=r+x$, $b=r-x$, so that

$$k^2=\frac{4xr}{(x+r)^2}=4\frac{x}{r}\left(1-2\frac{x}{r}+3\frac{x^2}{r^2}-4\frac{x^3}{r^3}+\cdot\cdot\cdot\right)$$

(10) becomes, on application of the usual logarithmic and binominal expansions,

$$\left.\begin{aligned} K=&\frac{1}{2}\log\frac{4r}{x}\left(1+\frac{x}{r}+\frac{1}{4}\frac{x^2}{r^2}+\frac{1}{4}\frac{x^3}{r^3}+\cdot\cdot\cdot\right)\\ &-\frac{1}{8}\frac{x^2}{r^2}-\frac{1}{8}\frac{x^3}{r^3}+\cdot\cdot\cdot\\ E=&\frac{1}{2}\log\frac{4r}{x}\left(2\frac{x}{r}-\frac{x^2}{r^2}+\frac{3}{2}\cdot\frac{x^3}{r^3}-\frac{9}{8}\frac{x^4}{r^4}+\cdot\cdot\cdot\right)\\ &+1-\frac{x}{r}+\frac{3}{4}\frac{x^2}{r^2}-\frac{x^3}{r^3}+\frac{153}{192}\frac{x^4}{r^4}+\cdot\cdot\cdot \end{aligned}\right\} \quad . \quad . \quad (11)$$

Applying (11) to (9)

$$m=\frac{16}{3}\pi r^3\left\{1+\frac{3}{4}\frac{x^2}{r^2}+\frac{3}{64}\frac{x^4}{r^4}+\ldots\right.$$
$$\left.-\frac{3}{2}\log\frac{4r}{x}\left(\frac{x^2}{r^2}-\frac{1}{8}\frac{x^4}{r^4}+\ldots\right)\right\} \quad . \quad . \quad . \quad . \quad (12)$$

In order to find the mutual induction between two semi-infinite coils, A and B, in contact, each having mean radius R, winding depth 2 X and n turns per unit area of section, divide the coils into cylindrical filaments of width δl and add the mutual inductions of the filaments as follows :—

Take the mutual induction between a filament of radius $r+x$ in coil A and one of radius $r-x$ in coil B. Associate with it the (same) mutual induction between the filament $r-x$ in A and $r+x$ in B. For these two pairs the sum of the mutual inductions is

$$\delta M=2n^2m(\delta l)^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

Let r vary by steps $\delta r=\delta l$ from $r=R-X+x$ to $r=R+X-x$ The sum of the expressions δM will then give the sum of the mutual inductions of all filaments at a distance $2x$ apart.

Perform a second summation with x varying by steps $\delta x=\frac{1}{2}\delta l$ from $x=0$ to $x=X$. This will give the required mutual induction $M(o)$. Hence

$$M(o)=4n^2\int_0^X dx\int_{R-X+x}^{R+X-x} m\,dr \quad . \quad . \quad . \quad . \quad (14)$$

Applying (14) to (12)

$$M(o)=\frac{64}{3}n^2\pi R^3X^2\left\{1+\frac{23}{48}\frac{X^2}{R^2}-\frac{1}{1600}\frac{X^4}{R^4}\right.$$
$$\left.-\log\frac{4R}{X}\left(\frac{1}{4}\frac{X^2}{R^2}-\frac{1}{80}\frac{X^4}{R^4}\right)\right\} \quad . \quad . \quad . \quad (15)$$

or putting $X=TR$, $2nX=N$, so that N is the number of turns per unit length in either coil, the correction $\Delta M(o)$ is given by

$$\frac{\Delta M(o)}{\pi^3N^2R^3}=-\frac{4}{3\pi}\left\{T^2\left(\log\frac{4}{T}-\frac{23}{12}\right)-\frac{T^4}{20}\left(\log\frac{4}{T}-\frac{1}{20}\right)\right\}. \quad (16)$$

7.

In order to see whether the effect of higher powers of T is appreciable, the mutual induction $M(o)$ will now be found by

another method. In a Paper on thick coaxial coils* the author has shown that the mutual induction N(o) between two semi-infinite coils of outer radii unity and r respectively, of zero inner radius, and with their ends in contact, is given by

$$\frac{N(o)}{2\pi^2 r^3}=\frac{1}{6}\left\{1-\frac{3}{20}r^2\left(\log\frac{4}{r}+\frac{9}{20}\right)+3r^4\sigma\right\}, \quad . \quad . \quad (17)$$

where $\sigma=\sum_0^\infty\left(\frac{1\cdot3\ldots2n+3}{2\cdot4\ldots2n+4}\right)^2\frac{r^{2n}}{(2n+7)(2n+3)(n+3)(n+1)}$, (17A)

the winding density of each coil being unity. When $r=1$ we have the alternative form

$$\frac{N_1(o)}{2\pi^2}=\frac{1}{30\pi}\left\{17-6\left(1-\frac{1}{3^2}+\frac{1}{5^2}-\frac{1}{7^2}+\ldots\right)\right\}$$

$$=0\cdot122063419. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18)$$

Further, the mutual induction M(o) between two similar hollow coils of external and internal radii unity and r is

$$M(o)=(1+r^5)N_1(o)-2N(o). \quad . \quad . \quad . \quad . \quad (19)$$

Since in (17) the general term is given, any desired accuracy may be obtained by calculating a sufficient number of terms, although when r is nearly equal to unity the method becomes tedious, as (19) is then the difference between two nearly equal quantities.

Table III. compares the values of M(o) as found by (17) and (19), and by (16). The second column is calculated by (17), the third by (19) and the fourth by (16). It is seen that the value of $N_1(o)$ is in good agreement with (18) and that formula (16) is extremely accurate for all winding depths less than one-quarter the radius of the coils.

TABLE III.

r.	N(o)/$2\pi^2r^3$.	M(o)/$2\pi^2$.	
		By (19).	By (16).
1·00	0·122063423	...	...
0·95	0·125119075	0·001965839	0·0019658390
0·90	0·12817228	0·007265462	0·0072654615
0·85	0·13120878	0·01506643	0·015066436
0·80	0·13421539	0·02462460	0·02462459

* Butterworth, "Phil. Mag.," p. 578, April, 1915.

8. *The Correction* $\Delta M(l)$.

It is shown in the Paper referred to in the preceding section that the mutual induction between two semi-infinite coaxial coils of outer radii unity and r, inner radius zero and at distance z apart is given by

$$N(z,r)=\frac{\pi^2 r^3}{z}\left\{\frac{1}{9}-\frac{1}{(2z)^2}\left(\frac{1}{15}+\frac{1}{15}r^2\right)\right.$$
$$+\frac{1}{(2z)^4}\left(\frac{2}{21}+\frac{6}{25}r^2+\frac{2}{21}r^4\right)$$
$$-\frac{1}{(2z)^6}\left(\frac{5}{27}+\frac{6}{7}r^2+\frac{6}{7}r^4+\frac{5}{27}r^6\right)$$
$$\left.+\ldots\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (20)$$

the winding density of each coil being unity and z being greater than $2r$.

From the laws of combination of mutual inductances and from dimensional considerations, the mutual induction between two similar hollow coils of outer and inner radii a and b, and at a distance c, is

$$M(c)=a^5\left\{N(z,1)-2N(z,r)+r^5N\left(\frac{z}{r},1\right)\right\}, \quad . \quad . \quad (21)$$

where $z=c/a$, $r=b/a$.

The relation (21) when applied to (20) converts the term r^{2n+3}/z^{2p+1} into $\frac{a^5}{z^{2p+1}}(1-2r^{2n+3}+r^{2p+6})$ and the term $r^{2p-2n+3}/z^{2p+1}$ into $\frac{a^5}{z^{2p+1}}(1-2r^{2p-2n+3}+r^{2p+6})$.

Since these terms have the same coefficient their sum is converted into

$$\frac{2a^5}{z^{2p+1}}(1-r^{2n+3})(1-r^{2p-2n+3})$$

or
$$\frac{2}{c^{2p+1}}(a^{2n+3}-b^{2n+3})(a^{2p-2n+3}-b^{2p-2n+3}). \quad . \quad (22)$$

Hence, if in (20) we replace $2r^{2n+3}/z^{2p+1}$ by the expression (22) we obtain the required series for $M(c)$. Making this

substitution, and in addition putting $a=1+T$, $b=1-T$, we find on arranging in ascending powers of T

$$\frac{cM(c)}{4\pi^2}=T^2\left(1-\frac{1}{2c^2}+\frac{5}{8c^4}-\frac{35}{32c^6}+\ldots\right)$$
$$+\frac{2}{3}T^4\left(1-\frac{7}{4c^2}+\frac{17}{4c^4}-\frac{775}{64c^6}+\ldots\right)$$
$$+\frac{1}{9}T^6\left(1-\frac{39}{10c^2}+\frac{507}{20c^4}-\frac{2167}{80c^6}+\ldots\right)$$
$$+\ldots, \quad \ldots\ldots\ldots \quad (23)$$

which holds for coils of mean radius unity, winding depth 2T, length c, and unit winding density. By multiplying $M(c)$ by n^2R^3 we get the formula for coils of mean radius R, winding depth 2TR, length cR, and winding density n. Finally, since $2n$TR$=$N, the number of turns per unit length

$$\frac{\Delta M(l)}{\pi^2N^2R^3}=\frac{2}{3}\frac{T^2}{c}\left(1-\frac{7}{4c^2}+\frac{17}{4c^4}-\frac{775}{64c^6}+\ldots\right)$$
$$+\frac{1}{9}\frac{T^4}{c}\left(1-\frac{39}{10c^2}+\ldots\right)$$
$$+\ldots. \quad \ldots\ldots\ldots \quad (24)$$

9.

As a check on (24), the value of $M(l)$ can be calculated by (24) and again by (20) and (21) directly. In getting $M(l)$ for the mean radius the formula

$$M(l)=\frac{\pi^2N^2R^3}{c}\left\{1-\frac{1}{2c^2}\right)-\frac{3\cdot5}{3\cdot4}\frac{1}{c^2}\right)-\frac{5\cdot7}{4\cdot5}\frac{1}{c^2}\right)\ldots\Big\}, \quad (25)^*$$

due to Havelock, may be used. The first term in (23) agrees with (25).

For example, with R$=9$, 2TR$=2$, cR$=40$, N$=200$,

(20) and (21) give $M=6{\cdot}454_3\pi^2$ millihenries.

(24) and (25) give $M=6{\cdot}4544_1\pi^2$ millihenries.

The agreement is satisfactory within the limits of accuracy of the formulæ.

* *See* footnote on p. 372.

10. *Comparison of the Corrections.*

Table IV. shows the results obtained by Rosa's, Cohen's and the present method for three coils, for which

$$c=4,\ 5,\ 6,\ ;\ T=1/10,\ 1/12,\ 1/10,\ \pi^2N^2R^3=1.$$

It is seen that Rosa's correction is 2 per cent. too high, while the total induction is one-fifth per cent. low.

In Cohen's method the tabulated values are for five layers. Although the results show very good agreement with the correct values, this is accidental, being due to the number of layers chosen. If a larger number of layers are taken the results get worse instead of better. Thus, for $c=4, T=1/10$, with

$m=$	1	2	3	4	5	10	infinity
L=	12·70,	12·11,	12·07,	12·06,	12·09,	12·14,	12·19

the correct value being 12·09.

TABLE IV.

c	4	5	6
T	1/10	1/12	1/10
ΔL_1 by (8)	−1·0133333	−1·0690236	−1·5200000
$\Delta M(c)$ by (16)	−0·0075138	−0·0057568	−0·0075138
$\Delta M(l)$ by (24)	+0·0015093	+0·0008676	+0·0010621
ΔL by (7)	−0·995287	−1·055775	−1·502848
ΔL by (6), Rosa	−1·0207	−1·0763	−1·5492
L_0 by (5)	13·090172	16·997069	20·933552
L present method)	12·094885	15·941294	19·430704
L (Rosa)	12·0695	15·9208	19·3843
L (Cohen) by (3) using 5 layers	12·091	15·939	19·462

11.

It is convenient to summarise the steps by which the inductance of a solenoid can be calculated. The coil is assumed to have a length greater than twice its diameter and a thickness of winding of less than one-tenth the diameter. The formulæ are sufficient to give four figure accuracy in the final result. The method is as follows :—

1. Calculate the inductance for the mean radius R without the end correction from the formula,

$$L_1=4\pi^2N^2R^3c,$$

where N is the number of turns per unit length (including all layers) and Rc is the coil length.

2. Apply the "end" correction to obtain the inductance

(L_2) for a current sheet of the mean radius of the coil from the formula,

$$L_2=L_1\left(1-\frac{8}{3\pi c}+\frac{1}{2c^2}-\frac{1}{4c^4}\right).$$

3. Apply the "thickness" correction to obtain the inductance (L_3) for a coil of winding depth 2TR from the formula,

$$L_3=L_2+\Delta L_2,$$

in which

$$\Delta L_2=-\frac{2}{3}L_1T\left\{1-\frac{T}{2}-\frac{T}{\pi c}\left(\log_e\frac{4}{T}-\frac{23}{12}\right)\right\}.$$

4. Apply the "insulation space" correction to obtain the true inductance L, for which *see* Rosa ("Bull." Bureau of Standards, 3, p. 37, 1907).

12. *Summary.*

1. It has been shown that the formulæ previously given for correcting for thickness in determining the inductance of solenoids fail to give an accuracy of one part in a thousand.

2. The true correction formulæ are obtained which are capable of giving eight figure accuracy without undue labour.

3. Simplified formulæ are also given which hold when only four figure accuracy is required.

APPENDIX.

As portions of the criticism of Rosa's and Cohen's formulæ depend on arithmetical accuracy in calculation, the arithmetical details for $c=4$, $T=1/10$ are appended.

(*a*) *Inductance of Mean Layer Solenoid.*

By Havelock's formula (5):

$$\begin{aligned}L_0&=16\{1-0\cdot21220659+0\cdot03125000\\&\qquad-\quad 97656+\quad 7630\\&\qquad-\quad 835+\quad 109\\&\qquad-\quad 16+\quad 3\}\\&=16\{1\cdot03132742\\&\qquad-0\cdot21319166\}\\&=16\times0\cdot81813576\\&=13\cdot090172.\end{aligned}$$

(*b*) *Correction* ΔL_1 (formula (8)).

$$\Delta L_1=-\frac{8}{3}\times4\times\frac{1}{10}(1-0\cdot05)=-\frac{30\cdot4}{30}=+1\cdot0133333.$$

(*c*) *Correction* $\Delta M(o)$ (formula (16)).

$$\Delta M(o)=-\frac{4}{3\pi}\{0\cdot01(3\cdot68888-1\cdot91667)-0\cdot000005(3\cdot69-0\cdot05)\}$$

$$=-\frac{4}{3\pi}\{0\cdot0177221-0\cdot0000182\}$$

$$=-0\cdot424413\times0\cdot0177039$$

$$=-0\cdot00751376.$$

(*d*) *Correction* $\Delta M(l)$ (formula (24)).

$$\Delta M(l)=\begin{Bmatrix}0\cdot0016667-0\cdot0001823\\+\quad 277-\quad 49\end{Bmatrix}+\begin{Bmatrix}0\cdot0000028\\-\quad 7\end{Bmatrix}$$

$$=0\cdot0015093.$$

(*e*) *Correction* ΔL (formula (7)).

$$\Delta L=\Delta L_1-2\Delta M(o)+2\Delta M(l)$$

$$=-1\cdot0133333+0\cdot0150275+0\cdot0030186$$

$$=-0\cdot9952872.$$

(*f*) *Rosa's Correction* (formula (6) and Table II.).

$$\Delta L=-\frac{8}{\pi}\times\frac{4}{10}(0\cdot6922+0\cdot3099)$$

$$=-0\cdot31831\times3\cdot2\times1\cdot0021$$

$$=-1\cdot0207.$$

(*g*) *Cohen's Formula* (3).

Using five layers.

$m=5.$ $\quad\delta k=0\cdot04.$

$k_1=0\cdot92$	$k_2=0\cdot96$	$k_3=1\cdot00$	$k_4=1\cdot04$
$k_1^2=0\cdot8464$	$k_2^2=0\cdot9216$	$k_3^2=1\cdot0000$	$k_4^2=1\cdot0816$
$(m-1)k_1^2=3\cdot3856$	$m(m-1)k_1^2=16\cdot9$		$m(m-1)k_2^2=16\cdot9$
$(m-2)k_2^2=2\cdot7648$	$(m-1)(m-2)k_2^2=11\cdot1$		$(m-2)(m-3)k_2^2=5\cdot5$
$(m-3)k_3^2=2\cdot0000$	$(m-2)(m-3)k_3^2=6\cdot0$		
$(m-4)k_4^2=1\cdot0816$	$(m-3)(m-4)k_4^2=2\cdot2$		
Sum $=9\cdot2320$	Sum $=36\cdot2$		Sum $=22\cdot4$

$$\frac{2+c^2}{\sqrt{4+c^2}}-\frac{8}{3\pi}=0\cdot9\sqrt{20}-\frac{8}{3\pi}=\begin{cases}4\cdot0249\\-0\cdot8488\end{cases}=3\cdot1761$$

$$\sqrt{k_1^2+c^2}-\frac{7}{8}k_1=\sqrt{16\cdot846}-0\cdot805=\begin{cases}4\cdot1045\\-0\cdot8050\end{cases}=3\cdot2995$$

$$1-\frac{k_1}{\sqrt{k_1^2+c^2}}=\frac{3\cdot1845}{4\cdot1045}=0\cdot775.$$

Substituting in formula (3)

$$L=\frac{4}{5}\times 3\cdot 1761+\frac{8}{25}(9\cdot 2320\times 3\cdot 2995 \\ -0\cdot 02\times 36\cdot 2\times 0\cdot 775-0\cdot 056)$$

$$=2\cdot 5409+0\cdot 32(30\cdot 461-0\cdot 617)$$

$$=2\cdot 5409+9\cdot 5501=12\cdot 0910.$$

ABSTRACT.

The existing formulæ for coils of this type—viz., those of Rosa and Cohen—are shown to be inaccurate, the error amounting to one-fifth of 1 per cent. for the best formula when the winding depth is one-tenth the diameter of the coil. For greater winding depths the error is larger. The inaccuracy in Rosa's formula is due to the neglect of curvature in correcting for thickness, while in Cohen's formula the error is due to the approximate method of development.

New formulæ are developed by methods which are free from such approximations, and which apply to any coil for which the length is greater than twice the diameter, and the winding depth is less than one-tenth the diameter. These formulæ are capable of giving eight-figure accuracy.

Simplified formulæ are also given which are suitable when only four-figure accuracy is required.

XXIX. *Precision Resistance Measurements with Simple Apparatus.* By E. H. Rayner, *M.A., National Physical Laboratory.*

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The object of this Paper is to give some hints as to the comparison of resistances to a higher degree of accuracy than is usually attempted (except in the case of the inter-comparison of standards and of platinum thermometry), but which is often required for the determination of the constancy and of the temperature coefficient of resistance alloys and in the accurate estimation of temperature by resistance methods.

Simple resistance thermometers of copper or iron, &c., ought to be capable of a sensitivity of 0·001°C. This implies measuring resistance changes of about 1 in 300,000. Where very small temperature differences are in question, such as the lowering of the freezing point of solutions, there would appear to be no difficulty in making up a simple Wheatstone bridge with one pair of opposite sides of copper and the other of manganin. The whole might with advantage be immersed in the liquid whose temperature is to be measured. This would avoid the troublesome corrections commonly required in resistance thermometry. Balance would be obtained by a high resistance shunt in parallel with one side of the quadrilateral. Such an arrangement ought to be able to detect changes of temperature of the order of a ten-thousandth of a degree, implying a corresponding change of 1 part in 2,000,000 or 3,000,000 in the resistance.

It is known that small variations in the composition of alloys very often materially alters not only their resistance, but also the rate of change of their resistance with temperature. An accurate determination of the temperature coefficient affords a very delicate criterion of the chemical and physical similarity of such alloys, which will in many cases be much more sensitive and quickly performed than any ordinary chemical analysis.

The addition of another significant figure to the accuracy attainable in any physical measurement has always provided a new tool for use in research, and in measurements of resistances high accuracy is particularly easy of attainment, and the results may often be correlated with other physical and chemical properties.

Measurements of this nature are often neglected, owing to the idea that specially accurate Wheatstone, Carey-Foster or Kelvin bridges are required. This is by no means the case, and the results of some little experience in precision resistance measurements with simple apparatus may be useful to others.

The subject has been dealt with by R. T. Glazebrook in various British Association Reports, among which may be mentioned that of 1883, and in an appendix to the Report of 1906 F. E. Smith gives a valuable discussion of the subject. In this Paper Mr. Smith deals with precision resistance measurements more in connection with the comparison of the standards under his charge than with regard to practical hints as to everyday work. Various publications of the Bureau of Standards Washington and of the Reichsanstalt in Instrumentenkunde have drawn attention to the most likely sources of error, especially in the measurement of low resistances. Some of these are given in the bibliography at the end of this Paper.

The use of Bridge methods only is discussed. Potentiometer methods requiring great steadiness of current, which are sometimes of considerable magnitude, require equipment not commonly available.

Though one can hardly claim much novelty in the principles underlying resistance measurements, little seems to have been written on the application of precision methods to other than the inter-comparison of standard resistances. From the number of inquiries on the subject a description of methods suitable to various circumstances ought to be useful.

Many electrical indicating instruments can be read to 1 part in 1,000, and this indirectly fixes the lower limit of accuracy of resistances which can be of any practical use. A resistance box with errors of 1 part in 1,000 would, in fact, be a continual trouble, and a few parts in 10,000 may be regarded as a reasonable limit for Wheatstone bridge and similar resistances, and 1 or 2 parts in 10,000 for standard resistances. Precision measurements may be conveniently defined as those in which an accuracy of 1 part in 10,000 or more is attained.

The measurements described in this Paper attain the accuracy of 1 in 10,000 for practically every case, and in favourable conditions 1 in 1,000,000, or more. The ordinary Wheatstone bridge with plug contacts is not suitable for measurements of this nature, as it does not allow of a sufficiently continuous variation in resistance. If further accuracy is attempted in the usual manner by increasing the ratio between

the bridge arms the arrangement becomes insensitive. As an adjunct to a number of other resistances, it, or some similar resistance variable by 0·1 ohm up to 10,000 ohms, will be essential.

In measurements of this nature it is much easier to follow the methods adopted and accuracy attainable if numerical examples are given, and the results of experimental measurements are reproduced for this purpose. These have all been made with a Broca galvanometer of 10 ohms resistance. One of 100 ohms would have been more suitable for practically every experiment quoted, and for many one of 1,000 ohms, so that considerable advantage can be obtained in this respect by choice of one of a more suitable resistance. Still, the Paper shows what can be done when somewhat handicapped in this way. Nature has kindly allowed, when using a galvanometer of resistance R on a Wheatstone bridge instead of one of the most suitable resistance nR, that the sensitivity varies very slowly for a considerable change in n.

Thus, as the galvanometer has a resistance 25 times as high or as low as the most suitable value the sensitivity is only reduced to 0·4 of the best obtainable.*

For satisfactory work a voltmeter or ammeter to measure the main current is essential. In all measurements the galvanometer circuit should be kept closed, and the current to the bridge reversed, and, for quick working, resistances should be non-inductively wound. If reversing gives a large kick due to inductivity, the galvanometer circuit may be opened momentarily to reduce the effect. It is necessary that the galvanometer should not be affected electrostatically, and arrangements may be necessary to prevent this trouble. If the galvanometer is of the moving coil type it may be found necessary to connect the magnet and case to one terminal of the instrument, and if necessary to insulate the whole instrument separately if it has not insulating feet.

If the galvanometer is of the moving magnet type it is desirable that there should be electrical conductivity from the magnets and damping vane through the suspension to one of the terminals in a similar manner. This necessitates silvering the suspension if of quartz or glass. The conductivity of a silk suspension, if not specially cleaned but left "dirty," seems to be sufficient to avoid the trouble.

* Schuster, *see* Bibliography.

A metallic shield such as tinfoil between the galvanometer coils and the moving part may be used in place of a conducting suspension. This shield, and the portion of the instrument supporting the moving system, should be at some fixed, such as "earth," potential. The point to bear in mind is that on suddenly applying or reversing the current the potential of the coils of the galvanometer is suddenly altered, and unless the moving part can also attain the new potential at the same instant, or be screened from it, electrostatic forces will come into play, causing a false deflection of the instrument.

It is seldom that resistance measurements of precision are required, except between resistances of simple numerical ratio, and such measurements may be divided into the comparison of nominally equal resistances, and of resistances whose ratio is represented by some simple numerical fraction.

Case. I.—Comparison of Nearly Equal Resistances.

This measurement shares with the comparison of nearly equal weights and lengths the position of being capable of being carried out with the highest accuracy attainable in any physical measurement. Differences of 1 part in a million may generally be easily detected between resistances if not less than 1 ohm. It is essential for satisfactory work that they should be capable of carrying a reasonable current.

Method I.—Comparison by Interchanging in Two Adjacent Arms of a Wheatstone Bridge.

The two resistances are made part of a Wheatstone Bridge, A, B, the other two sides, P, Q, being nearly equal resistances, which may or may not be of the same nominal value as A and B. In the latter case A and B should be in series as regards the supply current, as this allows of the largest current to pass through the bridge, and is, therefore, the most sensitive arrangement, and it is also the most convenient. Balance is obtained by shunting one of the resistances with a relatively high resistance. If the bridge is nearly balanced without a shunt, a very high shunting resistance, perhaps a hundred thousand or a million ohms, will be required which is very seldom obtainable. This difficulty can be got over by shunting

one of the other arms, say, Q, by any convenient resistance Q′, whose value is not required to be accurately known, so that the required shunt on P is of convenient dimensions. It has the advantage that the variable resistance is always connected to the same part of the bridge which conduces to the ease and

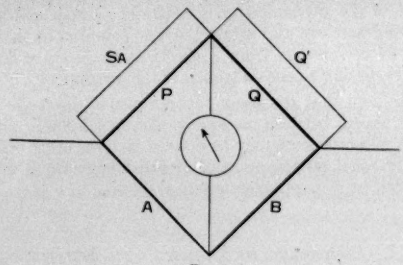

FIG. 1.

quickness of working. Thus, suppose that S_A is the shunting resistance required in parallel with P when A is connected to P. and S_B the shunting resistance when B is next to P.

Then $$\frac{A}{B}=1\div\frac{\left(\frac{1}{P}+\frac{1}{S_A}\right)}{Q} \text{ and also } \frac{A}{B}=\frac{Q}{1\div\left(\frac{1}{P}+\frac{1}{S_B}\right)}.$$

Eliminating Q

$$\frac{A^2}{B^2}=\frac{\frac{1}{P}+\frac{1}{S_B}}{\frac{1}{P}+\frac{1}{S_A}}=\frac{\left(1+\frac{P}{S_B}\right)}{\left(1+\frac{P}{S_A}\right)}.$$

If $\frac{P}{S_B}$ and $\frac{P}{S_A}$ are nearly equal and small compared with unity,

$$\frac{A}{B}=\frac{1+\frac{1}{2}\frac{P}{S_B}}{1+\frac{1}{2}\frac{P}{S_A}}=1+\tfrac{1}{2}P\left(\frac{1}{S_B}-\frac{1}{S_A}\right).$$

An example will make the accuracy obtainable more easily appreciated.

Thus P=Q=50 ohms, Q being shunted by a fixed resistance Q′ of about 9,000 ohms.

$A=B=20$ ohms. $S_A=8{,}572$ ohms. $S_B=8{,}493$ ohms.

$$\frac{A}{B}=1+\frac{50}{2}\left(\frac{1}{8{,}493}-\frac{1}{8{,}572}\right)=1+25\times 0{\cdot}000{,}001{,}08=1+0{\cdot}000{,}027.$$

Thus it will be seen that a difference of 80 ohms in the value of the two shunting resistances corresponds to a difference of 27 parts in a million in the ratio $\frac{A}{B}$, and as this difference may often be observable to 0·2 or 0·3 of an ohm, the ratio of A to B may be determined to a few parts in ten millions when A and B

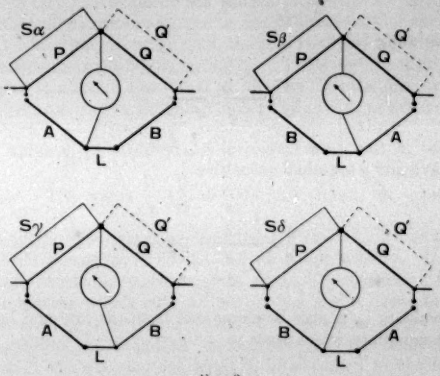

FIG. 2.

are nearly equal. It will be noted that absolute accuracy in P or in S_A and S_B is not required, only that the difference between S_A and S_B should not be seriously in error.

It is important that any leads connecting A and B to P and Q should remain fixed to P and Q, and should not be reversed when A and B are interchanged, otherwise any difference between them will be included in the measured difference between A and B.

In most cases a short connecting piece between A and B is necessary, in which case the method may be extended as follows: The galvanometer is first connected so that the connecting piece of resistance L is first added to B, and the ratio of $\frac{A}{B+L}$ found as above, after which the galvanometer connection is changed so that the ratio $\frac{B}{A+L}$ is obtained, four values of the shunting resistance being observed.

If the value of L is greater than the difference between A and B, then two equations of the form

$$\frac{A}{B+L}=1-x \text{ and}$$

$$\frac{B}{A+L}=1-y \text{ are obtained.}$$

Eliminating L

$$\frac{A}{B}=\frac{1-\frac{y}{2}}{1-y}\times\frac{1-x}{1-\frac{x}{2}},$$

and as x and y are small quantities

$$\frac{A}{B}=\frac{1-\frac{x}{2}}{1-\frac{y}{2}}=1+\tfrac{1}{2}(y-x).$$

To evaluate L, A may be eliminated from the first and last of these equations which leads to

$$\frac{L}{B}=\frac{\frac{1}{2}(y+x)}{(1-x)},$$

and as $1-x$ differs little from unity, and $\frac{L}{B}$ is not required to a high accuracy

$$\frac{L}{B}=\tfrac{1}{2}(y+x).$$

Example: A=B=50; P=Q=50. Permanent shunt Q′, of about 9,000 ohms on Q.

The diagrams, Fig. 2, show the connections

$$S_\alpha = 8{,}465{\cdot}4, \quad S_\beta = 8{,}691{\cdot}8, \quad S_\gamma = 8{,}653{\cdot}5, \quad S_\delta = 8{,}501{\cdot}5.$$

$$\frac{A}{B+L} = 1 + \frac{50}{2}\left(\frac{1}{8{,}691{\cdot}8} - \frac{1}{8{,}465{\cdot}4}\right) = 1 - 0{\cdot}000{,}769_2$$

$$\frac{B}{A+L} = 1 + \frac{50}{2}\left(\frac{1}{8{,}653{\cdot}5} - \frac{1}{8{,}501{\cdot}5}\right) = 1 - 0{\cdot}000{,}516_5$$

$$\frac{A}{B} = 1 + \tfrac{1}{2}(0{\cdot}000{,}051{,}6_5 - 0{\cdot}000{,}076{,}9_2) = 1 - 0{\cdot}000{,}012{,}6.$$

which gives the ratio to one part in 10 millions with good accuracy, and

$$\frac{L}{B} = \frac{L}{A} = 0{\cdot}000{,}064{,}3.$$

It is easy to estimate what value it is possible to put on the number of significant figures in the decimal part of the ratio A/B, which may be relied on in the final result by looking at the difference between S_α, S_δ, or S_β, S_γ. These are about 36 ohms correct to about 0·1 or 0·2 of an ohm, so that the final error in the decimal part ought to be under 1 per cent., in other words the value $\frac{A}{B} = 1 - 0{\cdot}000{,}012{,}6$ is correct to 1 unit in the last figure. The value $\frac{L}{A} = \frac{L}{B} = 0{\cdot}000{,}064{,}3$ must be constant during the experiment, so that being a short length of copper wire its temperature should not alter appreciably. In fact, in all resistance measurements of high precision it is the temperature which puts a limit to the useful accuracy, and constant temperature rooms and oil baths are necessary for the comparison of standard resistances. In the experiments quoted, the observations have to be repeated in the inverse order, and the mean taken to allow for temperature drift. In measurements of this nature each terminal block should have at least two, if not three, terminals in order that the movement of the galvanometer connection from one end of L to the other, and the connection of the high-resistance shunts, &c., may be made without disturbance of any contacts through which the main current passes.

In the example quoted A, B, P and Q were resistances of 50 ohms each wound on open frames. Each resistance was

composed of two circuits of 100 ohms each of No. 24 constantan wire, forming a non-inductive arrangement when the two are used in parallel. They are made so that 100 volts can be put on the frame so as to give 2 amperes, and are regularly used in connection with an accurate electrostatic voltmeter for measuring small alternating currents, a number of frames being put in series or parallel as required. The design is due to Mr. Campbell, and affords the most accurate method of measuring alternating currents of the order of 0·01 ampere to 20 amperes.

The winding is equivalent to two oppositely wound spirals, the current always being led in at one end of the frame and out at the other. If the more usual way of doubling a wire on itself were employed the inductance would be slightly less, but full voltage would exist between the adjacent wires near the ends. The wires carrying oppositely flowing currents are tied together, so as to reduce the inductance as far as possible, and the difference of potential between adjacent wires does not exceed about 3 volts. The frames are about $70 \times 50 \times 1{\cdot}5$ cm. in size, the grooves for the wires in the top and bottom bars being about 1 cm. apart.

It is not necessary to shunt the whole of one of the arms of the bridge by the variable resistance. It is often more convenient and accurate with the resistances available to shunt only a fraction.

The change of resistance with increasing current of one of the (50 ohms) frames capable of carrying 2 amperes, was determined in this manner. The wire for these resistances was specially selected as having a very small temperature coefficient. A bridge was made up of one of these frames as one arm and of nine others (three in series, three in parallel), so as to be equal in resistance in the adjacent arm, it being assumed that the heating in these being so much less would be negligible. The single resistance and the combination of nine were connected in series to a source of supply, so that somewhat over 200 volts could be applied. The other two arms were each of 10 resistances of 1,000 ohms (usable up to 1,000 volts). These resistances, high compared with the galvanometer resistance (10 ohms), rendered the arrangement insensitive at small currents, but the resistance of zero current can be calculated with good accuracy, as will be shown. One-tenth of the resistance Q (1,000 ohms) was shunted by a fixed resistance of about 5,000 ohms, and balance obtained by shunting a similar

section of P by a variable resistance of about the same value. The value of this resistance and the calculation of the relative resistance of P, and, therefore, of A at different currents, is shown in the following table :

Current.	R.	P.	δP.	Current $^{2\cdot6}$.	δP ÷ C$^{2\cdot6}$.
Ampere.					
0·98	5049·0	9834·68	...	0·95	...
1·41	5033·6	9834·26	−0·42	2·44	0·292
1·74	5016·0	9833·78	−0·90	4·22	0·281
1·98	4997·2	9833·26	−1·42	5·93	0·289
2·19	4978·0	9832·78	−1·90	7·68	0·285

The resistance P is the nominal value of the arm P as shunted

$$P = 9{,}000 + 1 \div \left(\frac{1}{1{,}000} + \frac{1}{R}\right),$$

δP is the change in P from its value at 0·98 ampere, and is

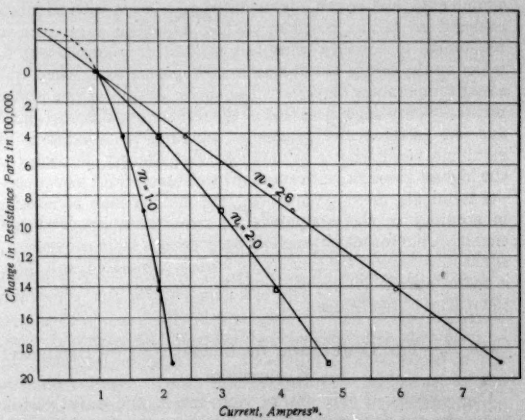

FIG. 3.

practically equivalent to parts in 10,000. From these values (assuming the bridge to be too insensitive for accurate readings below 1 ampere) it is impossible to determine accurately what would be the value of P at zero current by plotting δP against current (n=1) Fig. 3. It is much better to plot δP against the

square of the current which is nearly proportional to the temperature rise ($n=2$). This gives a line with some curvature in it. By trial it will be found that plotting δP against (current)$^{2\cdot6}$ ($n=2\cdot6$), a curve indistinguishable from a straight line results, leading to a value for δP at zero current of $+0\cdot28$. P for 2 amperes is $-1\cdot47$, so that the total diminution in resistance is 1·75 in 9,800, or practically 1·8 parts in 10,000 between the resistance measured at 2 amperes (100 volts) and that at a very small current.

Measurement of Equal Resistances by Substitution.

A very convenient method for the rapid comparison of a number of nearly equal resistances is to make up a Wheatstone bridge with one of them in one of the arms, and to substitute successively the various resistances in turn. The fine adjustment is conveniently made by shunting one of the arms by a high resistance. The other three resistances may be specially wound ones, and be roughly mounted on a board with all joints soldered as far as possible. When adjusting a number of resistances to equality using this method, it is convenient to make up the bridge out of lengths of the same wire, which will avoid temperature changes. The choice of dimensions of the resistances will depend on that of the resistances to be measured and the galvanometer resistance, enabling the conditions of maximum sensitivity to be obtained when desired. For work of the highest precision with standard resistances with heavy copper terminals, mercury cups, &c., the method may be inferior in accuracy to the comparison of two resistances by interchanging, but for many practical purposes it is more convenient, as the resistance of the connecting leads is eliminated, and there is more latitude in the choice of the resistances of the other three arms of the bridge.

CASE II.—THE COMPARISON OF RESISTANCES OF DIFFERENT MAGNITUDES.

Comparisons of this nature vary much, and only general principles can be suggested. To obtain a ratio of, say, 1 to 10, 10 resistances of the smaller value are put in series, each being capable of direct comparison with the standard. For the methods of building up and verifying resistances of less than 1,000 ohms the B.A. Paper of Mr. Smith should be consulted. Below 1,000 ohms special precautions have to be taken regarding the resistances of the end connections.

Verification of a Potential Divider.

An instructive example in the measurement of comparatively high resistances is that of a potential divider commonly used in conjunction with a potentiometer to measure 100 or 200 volts. A high resistance is connected across the supply voltage to be measured, and exactly 1 per cent. (or 0·5 per cent.) is tapped off and applied to the potentiometer. A voltage of 1·0 or 1·5 per 100 ohms is commonly used, so that for 100 or 150 volts the whole resistance will be 10,000 ohms, with a tapping at 100 ohms for the potentiometer. If standard resistances of 10,000 ohms and 100 ohms are available, the potential divider may be easily verified, provided the resistance of the internal connections can be neglected; but for precision work this is not sufficient. After being on the circuit for a short time it will be found that the effect of the heating is appreciable. If the whole box changed in resistance homo-

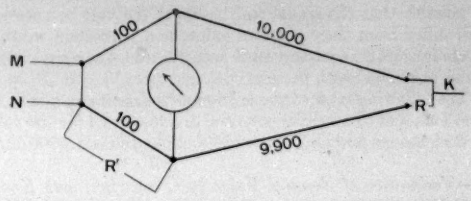

Fig 4.

geneously on this account it would not matter, but the 100-ohm section is often made of thicker wire for ease of adjustment, and even if this is not the case it may remain cooler than the rest, as it is practically always near one end of the resistance box. It is therefore desirable that the ratio of the two parts of the resistance, nominally, 1 : 99 should be determined at the working voltage. To do this we may take advantage of the useful theorem that the value of n nearly equal resistances connected in series is n^2 times their resistance in parallel to a very high order of accuracy. For instance, suppose half of them are as much as 1 in 1,000 high, and the other half 1 in 1,000 low, the n^2 law holds to one part in a million.

Suppose, then, we have 10 resistances of 1,000 ohms, which can be put in series or parallel, and also a resistance of 100 ohms, the 10 of 1,000 ohms in parallel can be compared with the

100-ohm resistance by inversion with very high accuracy. By putting the 1,000-ohm resistances in series, we now have two resistances of a nominal ratio of 100 : 10,000. It is assumed that these resistances are sufficiently liberally designed so as to be unaffected when 100 volts or more is applied to them in series. The volt box ratio to be proved is nominally 100 : 9,900, which may be converted for the purposes of measurement into 100 : 10,000 by two methods. The one is to add a variable resistance of about 100 ohms to the 9,900, making it up to 10,000, and the other is to shunt the 100-ohm section of the volt box with a variable resistance of about 10,000. This is the more delicate method, as 100 ohms change in this resistance is equivalent to 1 ohm in the first method. In the second method 10,000 ohms shunting the 100-ohm section of the volt box reduces the current in it by 1 per cent. The effect of this is neglected. Instead of resistances of 100 and $10\times1,000$ ohms, other convenient resistances of the same ratio can be used. It is possible that the actual resistances of the volt box sections will differ from their nominal value, and correction must be made for this if necessary, when using extraneous resistances in series or parallel with parts of it.

The following is an example from experiments on one of the best makes of commercial potential dividers rated for 150 volts.

Resistances, nominal, of the two sections 100 and 9,900 ohms.

I.—Verification of Nominal Value by Comparison with Known Resistance of 10,000 *ohms.*

Using the nomenclature of Fig. 1.

A=10,000 known. B=volt box. P=Q=50. Q′=4,000.
S_A=5,711. S_B=2,987.

$$\left(\frac{A}{B}\right)^2=\frac{2,987+50}{2,987}\times\frac{5,711}{5,711+50}=1\cdot00792.$$

$$\frac{A}{B}=1\cdot003,95.$$

The volt box, as a whole, is therefore low by nearly 40 ohms in 10,000.

II.—Determination of Auxiliary Ratio 100 : 10,000.

A resistance of 10 parts of 1,000 ohms each is arranged to be easily changed from series to parallel. Let F be the ratio of

the 100-ohm resistance to the resistance of the 10 of 1,000 ohms in parallel. From observations similar to those (Fig. 1)

$$F=1+\frac{50}{2}\left(\frac{1}{8,649\cdot2}-\frac{1}{8,734\cdot1}\right)=1+0\cdot000,028.$$

111.—Balancing the Auxiliary Resistance Against Volt Box.

(*a*) *By Addition of a Variable Resistance, R, of about* 100 *ohms.* The testing voltage is supplied at K and at M or N. If supplied at M the resistance of the short wire L is added to the 100 section of the volt box and is appreciable. The mean of the values, found when current is led in at M and N alternately, is taken. Thus, current led in at M, R=95·6; in at N, R=96·5; mean 96·05. This converted to volt box units ($\times1\cdot000,395$) =96·43. Then

$$\frac{A}{B}=\frac{F}{100}\times\left(1+\frac{R}{B}\right)=\frac{99\cdot969}{9,900}.$$

(*b*) *By Shunting the* 100*-ohm Section of the Volt Box.*—Current in at M, R′=10,176; current in at N′, R′=10,268; mean, 10,222; corrected for "volt box unit" 10,263.

$$\frac{A}{B}=\frac{F}{100}\times\frac{10,263+100}{10,263}=\frac{99\cdot969}{9,900}.$$

The two methods agree. These measurements were done with 10 volts.

After the application of 100 volts for half-an-hour the two methods gave $\frac{99\cdot959}{9,900}$ and $\frac{99\cdot960}{9,900}$, showing a change of 1 part in 10,000 due to the heating, the discrepancy between the two methods being 1 in 100,000.

The change of ratio which is caused by the application of 150 volts for 80 minutes, and the subsequent cooling when 10 volts is substituted is shown in Fig. 5.

The wire is wound round two parallel wooden cylinders separated such a distance apart that about half the total length is stretched in air between the cylinders, and the other half is in close contact with them. The free part very quickly takes up a certain temperature difference above the surrounding air. This accounts for the sudden change during the first minute. The temperature rise of the rest of the wire is

delayed by the heat capacity and conductivity of the wooden cylinders as indicated in the rest of the diagram.

It may be thought that measurements of the ratio of such an apparatus as a potential divider for 100 volts to an accuracy of the order of 1 part in 100,000 is far beyond the accuracy to which an apparatus can be usefully employed. It must, however, be remembered that it is one important link in the maintenance of the voltage of commercial supply networks at their correct value throughout the world.

Though it might not be generally expected, the accurate measurement of voltage is of prime importance in photometry. It is found that a series of photometric comparisons can be made to an accuracy of one part in a thousand. As the candle-power of a lamp varies four or five times as quickly as the

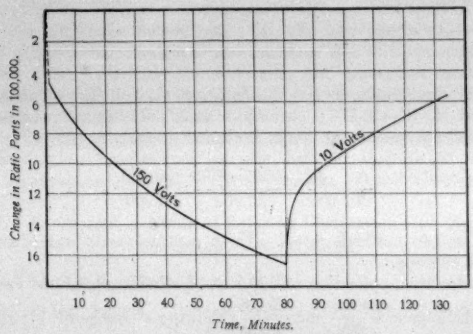

FIG. 5.

voltage, this amount would be accounted for by a change of 1 part in 5,000 in the voltage. It is, therefore, necessary that any errors in the electrical measuring instruments should be known to an accuracy of 1 in 10,000, and measurement of the magnitudes of these errors to the order of 1 in 100,000 is not superfluous.

A simple method of proving the ratio of such potential dividers under working conditions may be of use to makers, as the above results show that there is room for improvement in those at present obtainable.

Use of Continuous and Alternating Currents Superposed.

The author has shown in a Paper* read before the Institution of Electrical Engineers that useful information could be obtained by superposing a continuous potential of a few volts in series with an alternating one of several thousand when testing insulating materials, measuring the amount of continuous current passing, and also the alternating current and power.

The experiments on the volt box just described indicate the importance of measuring their ratios under the working voltage. As continuous voltages of 100 or more are not always so easily obtainable as alternating voltages, and still less is a voltage of 1,000, a similar superposition of a small measuring continuous voltage on a larger heating alternating voltage was tried on the same volt box. The continuous voltage was 13, and the alternating voltage 100. The rate of change of ratio after the application of the alternating voltage was practically the same as if 100 volts continuous were applied. Balance is first obtained with the continuous potential alone, before switching on the alternating potential. The heating effect of the 13 volts is negligible, compared with the 100 volts, the virtual voltage being $\sqrt{100^2+13^2}=100{\cdot}8$. With a moving needle galvanometer it is possible that any alternating current passing through it may affect it, but if a balance with continuous current is first obtained the alternating current will have an insignificant effect. If the time constants of the various parts of the circuit are notably different, which is not likely, as wire of high resistance per unit length is used for such resistances, some disturbance may be caused. This, however, is not likely to be of serious importance, as it is *change* of resistance for some time after application of the full rated voltage that is being looked for, due to rise of temperature, rather than instantaneous effect on switching on the alternating voltage due to differences of time constant.

Kelvin Double Bridge.

For the comparison of unequal resistances, especially when one or both is less than 1 ohm, the Kelvin double bridge cannot be surpassed.

In the comparison of equal resistances of the order of 50 ohms it has been seen that the short piece of wire connecting A and B

* E. H. Rayner, "High Voltage Tests and Energy Losses in Insulating Materials," "Journal of the Inst. of Electrical Engineers," Vol. 49, p. 47, 1912.

affects as much as the last three significant figures attainable in the ratio A : B. If it were practicable to connect the galvanometer exactly to the resistance centre of this wire, allowing for the uncertain contact resistances at the ends, then it would be unnecessary to double the number of observations which are required to eliminate its resistance, Fig. 6. The elegant method of surmounting this difficulty adopted in the Kelvin bridge is to connect the galvanometer, not to the middle point of the wire, but to the middle point of a resistance in parallel with it which is high in comparison with that of the joining wire, Fig. 7. The ends of this resistance should not be joined to the terminals through which the main current passes, but to a separate pair (the potential points of the resistance when these are provided). The resistance may be from 2 to 200 ohms, according to circumstances. In the general case where the two main resistances to be compared are not equal, this connecting resistance must be divided in the ratio of the main resistances.

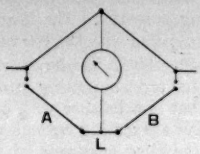

FIG. 6.

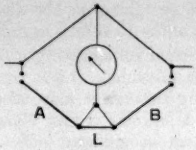

FIG. 7.

The resistances P, Q, are connected to the other potential points of the two resistances.

In commercial forms of the apparatus P is commonly made equal to R and Q equal to S, Fig. 8. Taking A to be a standard resistance, P and R may be of some simple value of about 20 to 100 ohms, and Q and S can be varied together from 0·1 to 1,000 ohms, and the ratio of B to A is read off at once to four figures. The sensitivity of the arrangement increases if low resistances, rather than high ones, are used for P, Q, R, S, but the resistance of the connecting leads makes 10 or 20 ohms the lower limit.

If the resistance of L is small compared with A and B, the values of R and S need not be so accurate as those of P and Q, and it becomes practicable to make R less than P, and S less than Q, with an increase in sensitivity without appreciable loss of accuracy. When measuring small resistances of 0·001

ohm or less, when L may be as large or larger than A or B, accuracy in R and S is nearly as important as in P and Q. The correctness of R/S may be verified by breaking the main circuit at L, when it will be seen that the system ought still to remain balanced. Practically the whole voltage of the source of supply will now come on to R and S in series, and care must be taken that these are not damaged if small resistances are used.

The equation representing the general condition that there be no current through the galvanometer is that

$$B=A\cdot\frac{Q}{P}+\frac{RL}{R+S+L}\left(\frac{Q}{P}-\frac{S}{R}\right).$$

As in the case of resistances of an ohm or more, precision

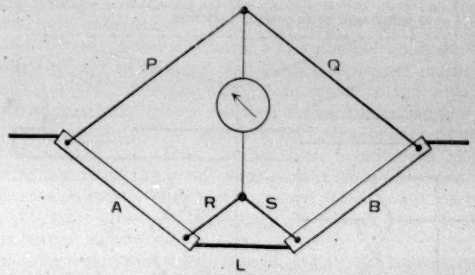

FIG. 8.

measurements are only required as a rule between resistances having a simple nominal ratio which enables the apparatus to be considerably simplified.

Special Form of Kelvin Bridge.

The necessity for apparatus of this nature for verifying the resistances used in precision alternating electrical measurements has led to the construction of a special Kelvin bridge for the purpose. The resistances to be verified vary from a few ohms to 0·001 ohm, and all have a simple numerical value. They are used in the current circuit when making alternating power and current measurements, and give a 2-volt drop of potential at their rated current.

The apparatus consists of two rows of resistances of 20 ohms each, there being 25 in each row. One row forms the part

P, Q, and the other R, S, as many of the resistances being used as required and connections are taken off for the galvanometer circuit.

As the resistances of the leads connecting the apparatus to the resistances to be compared cannot be made negligibly small, compared with 20 ohms, the end coils of the two rows have been adjusted to be 1 part in 1,000 low (0·05 ohm), and the leads used are adjusted to be 0·05 ohm.

These end coils must, therefore, always be used, and the bridge is adjusted to the correct simple ratio by short-circuiting

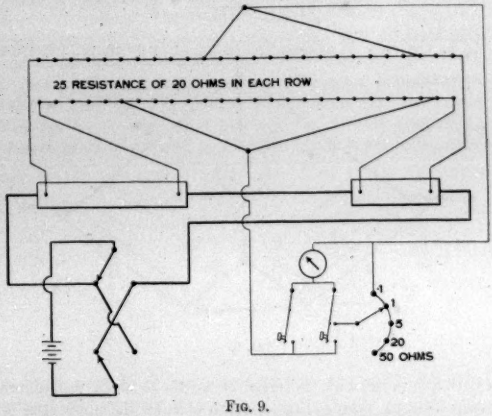

Fig. 9.

as many of the other coils as may be necessary. For convenience the two rows of 25 resistances are put in concentric circles, and copper straps are used for short-circuiting the necessary coils, these straps being also used for the galvanometer connections, which are arranged concentrically at the centre of the circle. The arrangement has the great merit that all the resistances have the same value, and so can be adjusted to equality with high accuracy. The absolute accuracy is quite immaterial. Balance is obtained by shunting A or B, whichever is relatively the higher.

The apparatus is capable of comparing any resistances of the ratio $\frac{m}{n}$, where m and n are any integers whose sum does

not exceed 25. Though it is possible to measure ratios of 24 : 1, lower ones such as 10 : 1 and 5 : 1 can be somewhat more accurately measured, as then at least two coils can be used in each of P and Q, and the small uncertainties of the resistances of the temporary connections, &c., become of less importance.

In the diagram the bridge is shown arranged for measuring a ratio of 5 : 1, P consisting of 10 coils, Q of 2, R of 5 and S of 1. It has been found satisfactory to mount the terminals directly on the wooden base which is soaked in hot wax. The holes through the wood are tapped, and the terminals screwed up tight, and locked by brass nuts in the usual way. This effectually prevents the terminals working loose, which is inevitable in wood if they pass through clearance holes.

The arrangement shown for the galvanometer keys has been found very satisfactory. When one key is depressed the galvanometer is first of all shunted by a resistance which can be varied from 0·1 to 50 ohms, and further depression makes a second contact connecting the galvanometer to the circuit. When approximate balance has been obtained the other key is depressed instead, which connects the galvanometer to the circuit without any shunt. It has been found much more satisfactory than the usual arrangement of a key with a resistance in circuit when half depressed which is cut out on applying more pressure. A mercury cup reversing key for the main current is provided.

In order to compare a resistance of, say, 0·001 ohm with one of 1 ohm it is necessary to do so in a series of steps, using intermediate resistances of, say, 0·1 and 0·01 ohm with a ratio of 10 : 1 in the bridge resistances every time. Any inequalities in the bridge resistances will cause a cumulative error in the final result, so that they should be very accurately adjusted to equality. These differences have been kept down to 2 or 3 parts in 100,000, so that in a "step-down" of 3 steps the accumulated error does not exceed 1 in 10,000.

Resistances of Incommensurable Ratios.

The accurate measurement of resistances of this nature is very seldom required. There is one case, however, in connection with the use of standard cells which is of practical importance. In apparatus of a potentiometer nature it is usual to adjust the current very exactly to some simple value, such as 0·01 ampere, and the easiest way to do so is to pass it through a resistance of 100 times the nominal value of the

voltage of a Weston cell ($100 \times 1{\cdot}0183$ ohms at 20°C.). If the potential divider has a number of equal resistances in series, each designed to have 10 volts on its terminals, it is a simple matter to connect a number of these, mostly in parallel, such that their resistance is exactly that to which the cell resistance has to be adjusted, which can therefore be done with great accuracy. Sixteen such resistances arranged in a certain manner will give a resistance equivalent to the voltage 1·0184, which is the voltage of Weston cell at 17°C. Other combinations may be made which will give slightly different values corresponding to the voltage of the cell at other temperatures. The subject has been dealt with in a previous publication.*

General Remarks.

The following points should receive attention in such measurements as have been described :—

1. The current should be as large as practicable, having regard to the change of resistance with temperature.

2. It should be measured by an ammeter.

3. In order to enable large currents to be used resistances should be liberally designed.

4. For ratio arms and other resistances frequently used it is of great advantage to wind them on open frames. The terminal blocks should have three terminals each. Two or three resistances of this nature of each of such values as 10, 100 and 1,000 ohms each will be very useful. They may be wound with two circuits so as to be very non-inductive.

5. Ten equal resistances which can be quickly changed from series to parallel are very useful. In order to avoid uncertainties due to contacts and differences of current paths in the two arrangements, 1,000 ohms each is desirable. These may be wound on micanite, which allows of better cooling than several layers on a coil. If heavy copper ends and mercury cups are used, 100-ohm units may be used instead of 1,000 ohms.

6. In Wheatstone bridge arrangements final balance is best obtained by a shunt of relatively high resistance in parallel with some part of the circuit. The higher the relative resistance the more delicate is the adjustment, and the less the importance of the accuracy of the shunting resistance. The system may very

* Paterson, Rayner and Kinnes, "The Use of the Electrostatic Method for the Measurement of Power," "Journal of the Inst. of Electrical Engineers," Part 221, Vol. 51, p. 315, 1913.

often be put intentionally out of balance to enable a suitable shunting resistance to be employed.

7. It is generally worth while, as far as possible, to design apparatus to be "self-checking," *i.e.*, in such a manner that the ratios of the various parts can be easily determined without the use of an external resistance whose value is required to be accurately known. Absolute accuracy of adjustment of resistances, though convenient, is not essential in most laboratory apparatus. It is only the relative proportions of the parts that are of importance in such instruments as potential dividers, potentiometers and Kelvin double bridges. It is only when resistances are used for the measurement of current that absolute accuracy becomes of importance.

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Other useful references will be found in the above Papers.

ABSTRACT.

The Paper describes methods by which the comparison of resistances can be made to an accuracy of 1 in 10,000 or higher by using simple apparatus usually available in electrical laboratories, or which can be easily constructed with little skilled assistance.

The comparison of nominally equal resistances of 1 ohm and upwards by the usual method of shunting one side of a nearly balanced quadrilateral by a high resistance is mentioned, and variations on this when only part of one resistance is shunted are often useful. The great advantages of having resistances capable of carrying comparatively large currents is illustrated, especially for measuring changes of resistance of commercial apparatus under working conditions. The determination of errors in a volt box for use with a potentiometer is described at some length. This is of especial importance in precision photometry.

If a sufficient continuous-current voltage is not available for testing such apparatus as high-potential dividers, it is shown that using sufficient continuous current to secure sensitivity the heating may be supplied by superposed alternating current.

Resistances in common use are very generally of simple numerical value, and a Kelvin bridge specially designed for the comparison of such resistances is described. It consists essentially of two rows of 25 resistances of 20 ohms each.

DISCUSSION.

Colonel SQUIER thought the Paper was very useful and likely to be generally helpful.

Mr. F. E. SMITH made several suggestions which, he thought, would improve the Paper. First, the conventional lettering of the arms of the bridge should be adopted, and an equation showing the sensitiveness of an ordinary bridge should be inserted. The Kelvin bridge arrangement could not be brought forward too prominently. If the equation representing the balancing condition was inserted it would be clear to all what errors might be introduced if the resistance of this lead was too high. With regard to the interchange method of comparing nominally equal resistances it was not the best or simplest for coils with potential leads. Suppose P, Q, R, and S (Fig. 10) each have a nominal resistance of 1 ohm, but that P is provided

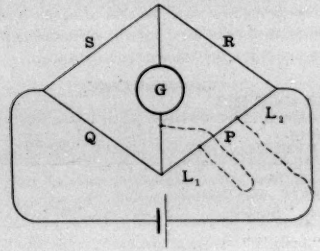

FIG. 10.

with potential leads, and it is the value of P between these which is desired. Let L_1 and L_2 be the resistance of the leads of P. Then, when the connections are as in full line we have

$$P+L_1+L_2=\frac{QR}{S'}, \quad \ldots\ldots\ldots\ldots \quad (1)$$

S′ being the shunted value of S necessary to obtain a balance.

By means of a switch the connections are now changed to those shown by the dotted lines, when

$$P=\frac{(Q+L_1)(R+L_2)}{S''}, \quad \ldots\ldots\ldots\ldots \quad (2)$$

S″ being the new shunted value of S.

Adding (1) and (2) and neglecting very small quantities we have

$$2P=QR\left(\frac{1}{S'}+\frac{1}{S''}\right).$$

The sensitiveness of this method is identical with the method of interchange, and it is much more convenient for coils with potential leads. The thermal E.M.F.s in a bridge, especially with coils containing Ni are often troublesome, and in the method of interchange the E.M.F. through the galvanometer is often reversed on interchanging. This should be remedied by reversing the galvanometer connection, or it may at times be remedied by reversing the position of a coil. Regarding heating effects, about three years ago, in collaboration with Dr. Glazebrook and Mr. Bousfield, I published some results on the increase of resistance with increase of current, but in no instance was any

difficulty found in obtaining the resistance for an infinite small current by plotting the change of resistance against C^2. Of course, all the curves were not straight lines. With resistance alloys such as constantan and manganin we do not expect them to be. The rise in temperature was, however, in all cases strictly proportional to C^2, and the temperature resistance curve could be plotted from the observation. Does the author think the constantan he worked with behaved differently, and if there was any real advantage in plotting change of R against $C^{1\cdot6}$ instead of C^2. Mr. Rayner's Paper will undoubtedly be of value to those who have to measure resistances with fair precision. There is, however, a bridge which for accuracy, ease of working and range (if suitably designed) is difficult to surpass. I refer to certain types of thermometer bridge. In potential, then, no matter what the resistance may be, between certain values, it has to be measured with very great accuracy. The bridge is readily calibrated, and requires no external apparatus for building up values. It is readily adapted to measure any resistance from about 0·1 ohm to 1,000 ohms, and the coils may have potential leads. I fear that because some bridges have been designed for precision work they are regarded as unsuitable for other work of a lower degree of accuracy. This fortunately is not true.

Mr. DUDDELL said that with regard to the building up of resistances he had in his laboratory a large number of 1,000 ω coils which were capable of carrying 0·1 ampere. By arranging these units in series and parallel combinations he could readily obtain any ratio he wished, and the building up of a potentiometer with these coils was very easy and convenient. An advantage in the use of high resistance coils such as 1,000 ω for work of this kind was the relative unimportance of the connection errors, which were very small if the resistances had substantial plug contacts.

Mr. A. CAMPBELL said that he had introduced the system of non-inductive winding on chanelled frames about 16 years ago, and it had been publicly in use in the National Physical Laboratory for many years. It appeared to have been patented recently by a foreign firm. The lessening of the inductance is obtained by winding a left and right-handed coil in the same set of channels and connecting them in parallel, tying the contiguous pairs of wires closely together. The capacity is well distributed if many channels are used. In a simpler method, which he has also used for many years, a single wire is twisted into a large number of loops, in each of which the wires are very close together, and each channel carries a single loop.

Mr. C. C. PATERSON said that it was inevitable when working with ordinary resistances on a high voltage that a good deal of heat should be dissipated. In potential dividers one makes the coils all of the same wire, measures the ratios at low currents and relies on these remaining the same at high currents. This is not the case, however, the ratio altering by an appreciable amount depending on the precise position of the portion from which the voltage is picked off—*e.g.*, at the end or the middle of the coils—on account of unequal temperature distribution. He instanced a case in which errors of two or three parts in 10,000 were introduced from this cause, and said that the method of superposing an alternating voltage on a small direct-current voltage was very useful for determining the ratios under the working conditions of heat dissipation.

Mr. D. OWEN welcomed a Paper on this subject. Emanating from the National Physical Laboratory, it would be regarded as embodying the soundest methods for use under the simple conditions which alone are generally attainable. The claim (in the first test quoted) of an accuracy of one part in 10 millions under simple conditions, when neither temperature nor the value of the testing current was specified, was misleading, and apt to propagate the mischievous illusion that *sensibility* of the bridge and

accuracy of the measurement were one and the same thing. The proof put forward of the formula for the method of interchange was, unfortunately, not rigorous, as the resistance of the leads at the outer ends of P and Q was not taken into account. The extension of the method which the author advocated, requiring four balances, appeared unnecessarily cumbrous; by connecting the galvanometer to a terminal fixed midway along the lead L the number of balances was reduced to two without loss of accuracy. In the test of effect of current strength on resistance of a coil the neglect of the heating effect in the adjacent arm was unjustifiable, as it amounted to 10 per cent. of the variation to be determined.

Dr. C. V. Drysdale (communicated): The two great principles upon which all standard bridges are based are those of Carey-Foster and of Kelvin, and in 1907 I combined both these principles in a bridge which is simple in construction, capable of working to the highest possible accuracy with resistances of any value, very rapid in use, and which gives the difference between the standard and test resistances directly in millionths of their value. This is when the two resistances to be compared are practically equal, and it is preferable and nearly always possible in a good laboratory at the present time to have a standard of approximately the same value as the resistance to be determined. In order, however, to arrange for stepping up or down, a self-contained ratio bobbin was devised embodying the principle laid down by Lord Rayleigh, having five coils of relative values 1, 3, 3, 3, 1. By suitable and convenient combination of these coils, which is now effected by solid bars and mercury contacts, not only can most of the uneven ratios required be obtained, but the arrangement is self-checking, and it forms an accurate series of standards built up from a single unit. It is true that when uneven ratios are employed, shunting has to be substituted for the slide-wire, but this is only necessary when comparing or adjusting a series of standards of different values, and it can be done with perfect facility on the same bridge. Apart from the convenience of adjustment, Mr. Rayner's bridge appears to suffer from the defects that contact errors in connecting his ratios are not eliminated, and that it is not self-checking. Except where the resistances have very appreciable and different time constants, also, it is decidedly bad practice to use a galvanometer key at all. Alterations of sensitiveness should be secured by varying the main current. Mr. Rayner correctly states that great advantage as regards contact errors can be secured without great loss of sensitiveness by keeping the resistance of the ratio coils moderately high, instead of attempting to fulfil the mathematical conditions for maximum sensitiveness. In practice it is possible to work to an accuracy of one or two parts in 10,000,000, when the ratio coils are never reduced below 10 or even 100 ohms.

Mr. Rayner replied as follows: As to lettering the different arms of the bridge P, Q, R, S, which Mr. Smith says is the conventional manner, neither the Americans nor the Germans seem to have adopted a conventional notation, and I have been troubled in following the discussion of various methods by the difficulty in keeping in mind the significance of the main and secondary resistances when P, Q, R and S are used. As to the change in resistance varying as the 2·6th power of the current, Mr. Smith considers the effect is most likely due to the non-linearity of the temperature coefficient rather than divergence from Newton's law of cooling, which his experiments on resistance thermometers seem to show hold with great exactness. Mr. Owen considers the neglect of an equal amount of heat generated in the nine other frames may account for it. I do not think it would make much difference, as the change of resistance at such a small temperature rise is very small. The point is not of great importance, as in practice a slight draught of air will make a considerable difference. The experiment is only quoted as an example of a method, and to give an idea what the effect may be in the case of resistances of a certain design intended for a much higher voltage and current than is usually employed. Mr. Smith has drawn my attention to the

fact that the statement in the description of Fig. 1 that for greatest sensitiveness resistances of similar values should always be put in series is incorrect. In practice, however, as resistances of different denominations are usually designed to withstand the same watts and not to carry the same current, the arrangement in the Paper is nearly always preferable. He also states that on reversing an equal armed bridge the potential of the galvanometer does not vary. This is the case if the reversing switch in the battery circuit connects directly to the bridge. If there is a regulating resistance for adjusting the main current added between the reversing switch and the bridge, then the potential of the galvanometer circuit is changed. For this reason an adjusting resistance should be put on the supply side of the reversing switch. I agree with Mr. Duddell as to the size of unit to be used when resistances are to be put in series or parallel. They should not be less than 1,000 ohms unless errors are carefully guarded against. Regarding the errors due to change of ratio of potential dividers mentioned by Mr. Paterson, the application of the method of superposing an alternating on a continuous potential for the accurate measurement of the effect was largely his suggestion. I agree with Mr. Owen that a sensitiveness to such a figure as, say, 1 in 10,000,000, does not necessarily imply similar accuracy of measurement, and the proof of the full use of the sensitivity lies in reproducing the numerical value on repeating the measurement. All the examples given have been reproducible in this sense. This does not mean that identically the same shunting resistances have been required on repeating a series of measurements, as very small temperature changes in a few minutes will affect the result; but it does mean that if, for instance, on repeating a given experiment, a different shunting resistance is required in the one position, an exactly equivalent change is found in the other position, showing that the measurement takes full advantage of the sensitivity available. There is all the difference between the measurement of small differences of temperature to a small fraction of a degree, such as has been suggested in the Paper, and the measurement of temperature generally to the corresponding fraction of a degree. Platinum silver, perhaps, one of the most stable materials for use as a permanent standard, is quite unsuitable as a working resistance by reason of its comparatively large temperature coefficients. Manganin and constantan can usually be obtained with a temperature coefficient of one-tenth the value. The suggestion of connecting the galvanometer to the middle of L in Fig. 2 is impracticable by reason of the resistance of the end contacts. Dr. Drysdale in his communication suggests the use of a combination of Carey-Foster and Kelvin bridges, using a bridge wire for the fine adjustment. It could hardly be termed a simple apparatus which could be made in any laboratory workshop, such as has been the aim of the Paper to describe. With bridge wire methods the thermoelectric force generated at the point of contact of the galvanometer circuit and the bridge wire is liable to cause trouble. It will vary in intensity when the contact is moved, and if contact with the wire is maintained to mask the effect some wear takes place. If the contact is broken before movement takes place the galvanometer moves from its false zero and must come to rest before the main current can be switched on. These and other difficulties have caused Mr. Smith to discard such methods in favour of shunting methods, in which the galvanometer circuit is always closed. The precision measurement of certain resistances of high and low values has been forced on us by the continually increasing demands for accuracy in technical electrical measurements, and the methods which have been described in the Paper have been adopted for some of these largely because they could be satisfactorily used to the accuracy required with the simple apparatus available. For a higher accuracy than that comtemplated in this Paper reference should be made to the papers named in the bibliography. Dr. Drysdale's series of articles in "The Electrician" give a most valuable account and criticism of standard resistances.

XXX. *On Electrically-maintained Vibrations.* *By* S. Butterworth, *M.Sc., Lecturer in Physics, School of Technology, Manchester.*

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1. In a Paper published some time ago by the author,* it was shown that when a vibrating system of one degree of freedom was set in motion by the action of a current, then, if the forces acting are proportional to the current, the system behaves as a parallel combination of a capacity, a conductance, and an inductance.

In some experiments arising from the theory, use was made of a wire carrying alternating currents, and stretched between the poles of an electromagnet. This system does not conform to the type discussed in that it enjoys an infinite number of degrees of freedom. It is proposed in the present Paper to extend the theory so as to include the possibility of a large number of degrees of freedom.

The only limitations imposed on the motion of the systems considered will be those usually assumed in the theory of small motions—viz., that the displacements from a position of stable equilibrium are so small that their squares may be neglected, and that the frictional forces are proportional to the velocities.

Two classes of displacing forces will be considered :—

(A) Those that arise when a coil carrying currents is in the neighbourhood of permanently magnetised bodies. The mechanical force at any point is then proportional to the current in the coil.

(B) Those due to the action of a charged condenser on electrified bodies. The mechanical force is then proportional to the E.M.F. across the plates of the condenser.

2. If the system possesses only one degree of freedom, its equation of motion is

$$(\alpha D^2+\beta D+\gamma)y=Y, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where $D=\frac{d}{dt}$, α, β, γ are the constants of inertia, damping and restoration respectively, y is the displacement, and Y the displacing force.

For systems of class A,

$$Y=AI, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

* Butterworth, "Proc." Phys. Soc., XXVI., p. 264, 1914.

where I is the current through the coil. Hence, in this case, the power to be supplied to maintain the motion is $AIDy$, or the E.M.F. due to the motion is

$$e=ADy. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Eliminating y from (3) by means of (1) and (2),

$$e=\mu I, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

where $$\mu=A^2/(aD+\beta+\gamma/D) \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

Hence, if r and l are the resistance and inductance of the coil and E its terminal E.M.F.,

$$E=(\mu+r+lD)I. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

If the motion of the system be prevented, and we place in series with the coil a parallel combination of a capacity (C_a), a conductance (S_a), and an inductance (L_a), the new terminal E.M.F. is

$$E'=(\mu'+r+lD)I, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

where $$\mu'=1/(C_aD+S_a+1/L_aD), \quad . \quad . \quad . \quad . \quad (8)$$

(6) and (7) are identical if $\mu'=\mu$, that is

$$C_a=a/A^2,\ S_a=\beta/A^2,\ L_a=A^2/\gamma. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

For systems of Class B,

$$Y=BE, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

where E is the E.M.F. across the plates of the condenser. Hence, in this case the power to be supplied to maintain the motion is $BEDy$, or a current

$$i=BDy \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

must be supplied in addition to that maintaining the charges on the condenser plates. Eliminating y from (11) by means of (1) and (10),

$$i=\nu E, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

where $$\nu=B^2/(aD+\beta+\gamma/D). \quad . \quad . \quad . \quad . \quad . \quad (13)$$

Hence, if s and c are the conductance and capacity of the condenser, and I the total current,

$$I=(\nu+s+cD)E. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

If the motion of the system be prevented, and we place in parallel with the condenser a series combination of an induc-

tance (L_b), a resistance (R_b) and a capacity (C_b), the new total current is

$$I'=(\nu'+s+cD)E, \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

where $$\nu'=1/(L_bD+R_b+1/C_bD), \quad . \quad . \quad . \quad . \quad (16)$$

(14) and (15) are identical if $\nu'=\nu$, that is,

$$L_b=a/B^2,\ R_b=\beta/B^2,\ C_b=B^2/\gamma. \quad . \quad . \quad . \quad . \quad (17)$$

3. We now develop the corresponding theory when the system enjoys m degrees of freedom.

Let the generalised co-ordinates be $y_1,\ y_2 \ldots y_m$, and the generalised velocities be $\dot{y}_1,\ \dot{y}_2 \ldots \dot{y}_m$. Then, for small motions about a position of stable equilibrium, the kinetic energy (T), the dissipation function (F), and the potential energy (V) are given by

$$\left.\begin{aligned} 2T&=\Sigma a_{rr}\dot{y}_r^2+2\Sigma a_{rs}\dot{y}_r\dot{y}_s \\ 2F&=\Sigma \beta_{rr}\dot{y}_r^2+2\Sigma \beta_{rs}\dot{y}_r\dot{y}_s \\ 2V&=\Sigma \gamma_{rr}y_r^2+2\Sigma \gamma_{rs}y_ry_s. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (18)$$

Using Lagrange's method we obtain the m equations of motion, of which the rth is

$$\frac{d}{dt}\left(\frac{dT}{d\dot{y}_r}\right)-\frac{dT}{dy_r}+\frac{dF}{d\dot{y}_r}+\frac{dV}{dy_r}=Y_r, \quad . \quad . \quad . \quad (19)$$

where Y_r is the generalised force corresponding to the co-ordinate y_r.

Let the co-ordinates be so chosen that F and V reduce to sums of squares, so that

$$\beta_{rs}=\gamma_{rs}=0,$$

and substitute the values of T, F, and V in equations (19). The rth equation now becomes

$$Y_r=(a_{rr}D^2+\beta_{rr}D+\gamma_{rr})y_r+\Sigma a_{rs}D^2y_s, \quad . \quad . \quad . \quad . \quad (20)$$

in which $D=\dfrac{d}{dt}$.

For systems of class A the displacing force at any point is proportional to the current, so that

$$Y_r=A_rI. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (21)$$

The power to maintain the motion is then given by ΣY_rDy_r, or the E.M.F. due to the motion is

$$e=\Sigma A_rDy_r=\Sigma e_r \text{ (say)}, \quad . \quad . \quad . \quad . \quad . \quad (22)$$

where $$e_r=A_rDy_r. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (23)$$

Eliminating y_r by means of (23) in (20), and using (21),

$$A_r I=(a_{rr}D+\beta_{rr}+\gamma_{rr}/D)\frac{e_r}{A_r}+\Sigma a_{rs}De_s/A_s. \quad . \quad . \quad . \quad . \quad . \quad (24)$$

If a_{rs} were zero the equivalent electrical system would be obtained by combining in series m parallel combinations of capacity, conductance and inductance, the values of which for the rth combination would be

$$C_{rr}=a_{rr}/A_r^2,\ S_{rr}=\beta_{rr}/A_r^2,\ L_{rr}=A_{rr}^2/\gamma_{rr}, \quad . \quad . \quad . \quad (25)$$

for this combination would satisfy the current-E.M.F. equations (24) and the relation $e=\Sigma e_r$.

When a_{rs} is not zero, the series arrangement of condensers must be replaced by a condenser network. The equivalent electrical system is then built up as follows: Connect $m+1$ points $P_0, P_1..P_m$ by condensers such that every point is connected to every other point once and once only. $m(m+1)/2$ condensers will be necessary. Let K_{rs} be the capacity of the condenser connecting P_r to P_s. Join successive points (P_{r-1}, P_r) by parallel combinations of conductance and inductance (S_{rr}, L_{rr}). Let a current I enter the system at P_0 and let it leave at P_m.

Then, if $i_1,..i_{r+1}—i_r,..—i_m$ are the currents entering the condenser network at $P_0,..P_r,..P_m$ respectively, i'_r is the current through the conductance S_{rr}, i''_r is the current through the inductance L_{rr}, we have

$$I=i_1+i'_1+i''_1=i_2+i'_2+i''_2=....=i_m+i'_m+i''_m.... \quad . \quad (26)$$

Also, if v_r is the potential of P_r

$$\left.\begin{aligned} i_1&=K_{01}D(v_0—v_1)+K_{02}D(v_0—v_2)+....\\ i_2—i_1&=K_{10}D(v_1—v_0)+K_{12}D(v_1—v_2)+....\\ &.....\end{aligned}\right\}$$

in which $K_{rs}=K_{sr}$.

Replacing the actual potentials by P.D.s between successive points (that is, putting $v_{r-1}—v_r=e_r$) and solving for $i_1, i_2..$,

$$\left.\begin{aligned} i_1&=C_{11}De_1+C_{12}De_2+....\\ i_2&=C_{21}De_1+C_{22}De_2+....\\ &.....\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (27)$$

in which $C_{rs}=C_{sr}$.

Also, $i'_r=S_{rr}e_r$, $i''_r=e_r/L_{rr}D$. . . . . (28)

Adding equations (28) to the rth equation of (27) we have from (26)

$$I=(C_{rr}D+S_{rr}+1/L_{rr}D)e_r+\Sigma C_{rs}De_s. \quad . \quad . \quad . \quad . \quad (29)$$

Again, if e is the total P.D. between P_0 and P_m,

$$e=\Sigma e_r. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (30)$$

A comparison of equations (29) and (30) with (24) and (22) shows that the electrical system is equivalent to the mechanical system provided that (25) holds, and

$$C_{rs}=\alpha_{rs}/A_rA_s. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (31)$$

For systems of class B, the displacing force is proportional to the E.M.F. (E) across the plates of a condenser, so that in (20)

$$Y_r=B_rE. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (32)$$

The power to maintain the motion is, as before, ΣY_rDy_r, so that by (32) the current * to be supplied to the condenser in order to maintain the motion is

$$i=\Sigma B_rDy_r=\Sigma i_r \text{ (say)}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (33)$$

where

$$i_r=B_rDy_r. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (34)$$

Eliminating y_r in equation (20) by means of (34) and using (32),

$$B_rE=(\alpha_{rr}D+\beta_{rr}+\gamma_{rr}/D)i_r/B_r=\Sigma\alpha_{rs}Di_s/B_s. \quad . \quad . \quad . \quad . \quad . \quad (35)$$

The equivalent electrical system follows immediately from (33) and (35). It is simply m series combinations of inductance, resistance and capacity arranged in parallel, and possessing mutual inductance between the various coils. If L_{rr}, R_{rr}, C_{rr} are the values of the inductance, resistance and capacity for the rth combination, and L_{rs} is the mutual induction between L_{rr} and L_{ss}, the conditions for equivalence are

$$\left.\begin{aligned} L_{rr}&=\alpha_{rr}/B_r^2, \quad R_{rr}=\beta_{rr}/B_r^2 \\ C_{rr}&=B_r^2/\gamma_{rr}, \quad L_{rs}=\alpha_{rs}/B_rB_s \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (36)$$

4. A stretched wire in a transverse magnetic field affords the simplest illustration of a system of class A enjoying an infinite number of degrees of freedom.

Instead of following the general method of section 3, the normal equations will be derived from the partial differential equation of motion.

* This current exists in addition to the charging and leakage currents.

Let the wire be fixed at two points at distance $2l$. Take the centre of the wire as origin and the undisplaced position of the wire as the axis of x. If the intensity of the magnetic field at x is H and a current I flows through the wire, the equation of motion is

$$m\frac{\partial^2 y}{\partial t^2}+\rho\frac{\partial y}{\partial t}=T_1\frac{\partial^2 y}{\partial x^2}+HI, \quad . \quad . \quad . \quad . \quad . \quad (37)$$

in which T_1 is the tension, m the linear density, and ρ the damping per unit length. The end conditions are

$$y=0 \text{ when } x=-l, \text{ and } x=+l \quad . \quad . \quad . \quad (38)$$

Whatever the form of the wire when displaced we may expand y in a Fourier series which holds from $x=-l$ to $x=+l$. Thus, let

$$y=\sum_{r=1}^{r=\infty}(y_r \sin a_r x+z_r \cos b_r x), \quad . \quad . \quad . \quad (39)$$

a_r, b_r are determined by the conditions (38) giving

$$a_r=r\pi/l, \; b_r=(2r-1)\pi/2l, \quad . \quad . \quad . \quad . \quad . \quad (40)$$

in which r is an integer.
H may be expressed in a similar series, viz.,

$$H=\sum_{r=1}^{r=\infty}(H_r \sin a_r x+K_r \cos b_r x). \quad . \quad . \quad . \quad (41)$$

Substituting (39) and (41) in the equation of motion (37)

$$\sum_{r=1}^{r=\infty}\{(m\ddot{y}_r+\rho\dot{y}_r+T_1 a_r^2 y_r-H_r I)\sin a_r x \cos b_r x\}$$
$$+(m\ddot{z}_r+\rho\dot{z}_r+T_1 b_r^2 z_r-K_r I)\cos b_r x=0, \quad . \quad . \quad (42)$$

whence the normal equations of motion are

$$\left.\begin{aligned}(mD^2+\rho D+T_1 a_r^2)y_r&=H_r I\\(mD^2+\rho D+T_1 b_r^2)z_r&=K_r I\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (43)$$

Before we can apply the results of section 3, it may be necessary to multiply these equations by a constant factor. To find this factor it is only necessary to form the expression for the kinetic energy.

We have
$$2T=m\int_{-l}^{l}\dot{y}^2 dx$$
$$=ml\sum_{r=1}^{r=\infty}(\dot{y}_r^2+\dot{z}_r^2)$$

by (39) and (40).

Referring to (18), $a_{rr}=ml$, so that (43) is identical with (20) if we multiply the former by l. We therefore have

$$\left.\begin{aligned} &a_{rr}=ml,\ \beta_{rr}=\rho l \\ &\gamma_{rr}=T_1 a_r{}^2 l \text{ when } A_r=H_r l \\ &\gamma_{rr}=T_1 b_r{}^2 l \text{ when } A_r=K_r l \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (44)$$

$$a_{rs}=0.$$

The values of H_r and K_r are determined by the form of H when expressed as a function of x; if H is an even function of x, H_r vanishes, if an odd function K_r vanishes.

For example, suppose H to be zero when $x>a$ and $x<-a$, and H to be constant when $a>x>-a$. Then by the usual method,

$$H_r=0,\ K_r=4H \sin (2r-1)\pi a/2l/(2r-1)\pi \quad . \quad . \quad (45)$$

5. Returning to the case of one degree of freedom, a system of class A behaves as a parallel combination of capacity, conductance and inductance, in series with an inductive coil, while a system of class B behaves as a series combination of inductance, resistance and capacity in parallel with a condenser.

The electrical equivalent of system A is unrealisable owing to the absence of resistance in the inductance, but the electrical equivalent of system B can be realised if the damping is sufficiently large. Moreover, a system of class B may be made to balance a system of class A on a Wheatstone bridge by placing the two systems in the conjugate arms (Q, R) and pure resistances (S, P) in the other arms.

The condition of balance is then by (6) and (14)

$$\frac{\mu+r+l\mathrm{D}}{\nu+s+c\mathrm{D}}=\mathrm{SP},$$

so that by (8) and (16) the balance will be independent of the form of the current if

$$L_b/C_a=R_b/S_a=L_a/C_b=r/s=l/c=\mathrm{SP}.$$

Hence for large dampings the constants of system A can be found by balancing with the electrical equivalent of system B.

This method may be extended to the general case. The resistance operator (μ) of system A is obtained by eliminating the m E.M.F.s (e_1, e_2 . . . . e_m) from the $m+1$ equations (29) and (30). This gives

$$-\Delta_1 \mathrm{I}=\Delta'_1 e,$$

where

$$\Delta_1=\begin{vmatrix} 0 & 1 & 1 & 1 & .. \\ 1 & k_{11} & k_{12} & k_{13} & .. \\ 1 & k_{21} & k_{22} & k_{23} & .. \\ 1 & k_{31} & k_{32} & k_{33} & .. \\ .. & .. & .. & .. & .. \end{vmatrix} \qquad \Delta'_1=\begin{vmatrix} k_{11} & k_{12} & k_{13} & .. \\ k_{21} & k_{22} & k_{23} & .. \\ k_{31} & k_{32} & k_{33} & .. \\ .. & .. & .. & .. \end{vmatrix}$$

in which

$$k_{rr}=C_{rr}D+S_{rr}+1/L_{rr}D$$

$$k_{rs}=C_{rs}D=k_{sr}.$$

Hence

$$r=-\Delta_1/\Delta'_1.$$

Similarly for system B the resistance operator is

$$1/v=-\Delta'_2/\Delta_2,$$

where

$$\Delta_2=\begin{vmatrix} 0 & 1 & 1 & 1 & .. \\ 1 & l_{11} & l_{12} & l_{13} & .. \\ 1 & l_{21} & l_{22} & l_{23} & .. \\ 1 & l_{31} & l_{32} & l_{33} & .. \\ .. & .. & .. & .. & .. \end{vmatrix} \qquad \Delta'_2=\begin{vmatrix} l_{11} & l_{12} & l_{13} & .. \\ l_{21} & l_{22} & l_{23} & .. \\ l_{31} & l_{32} & l_{33} & .. \\ .. & .. & .. & .. \end{vmatrix}$$

in which

$$l_{rr}=L'_{rr}D+R'_{rr}+1/C'_{rr}D$$

$$l_{rs}=L'_{rs}D=l_{sr},$$

the accents being introduced to distinguish system B from system A.

The condition for a complete balance when the two systems are in conjugate arms of a Wheatstone bridge is as before that μ/ν shall be independent of D. This will be the case if

$$\frac{L'_{rr}}{C_{rr}}=\frac{R'_{rr}}{S_{rr}}=\frac{L_{rr}}{C'_{rr}}=\frac{L'_{rs}}{C_{rs}}=SP.$$

The constants of a system of class B can be found by balancing with its equivalent electrical system in the adjacent arm of a Wheatstone bridge.

6. The balances discussed in the previous section are mainly of theoretical interest only. If we use alternating currents of frequency $p/2\pi$, then the measurements may be considerably simplified.

For a system of class A and of one degree of freedom, if the natural frequency of the vibrating system is $p_0/2\pi$ so that

$$p_0^2=1/L_aC_a=\gamma/a, \quad . \quad . \quad . \quad . \quad . \quad . \quad (46)$$

then when there is no damping and $p<p_0$ the system behaves as an inductance

$$L_0=L_a/(1-n^2), \quad . \quad . \quad . \quad . \quad . \quad . \quad (47)$$

in which $n=p/p_0$,

and as a capacity

$$C_0=C_a(1-1/n^2) \quad . \quad . \quad . \quad . \quad . \quad . \quad (48)$$

if $p>p_0$.

These results follow from (8) with $S_a=0$. If S_a is not zero, then we must suppose S_a to be placed in parallel with L_0 or C_0. In this case the moving system behaves as an inductance

$$L'_0=L_0/(1+p^2L_0^2S_a^2) \quad . \quad . \quad . \quad . \quad . \quad . \quad (47a)$$

in series, with a resistance

$$R'_0=p^2L_0^2S_a/(1+p^2L_0^2S_a^2) \quad . \quad . \quad . \quad . \quad (47b)$$

when $p<p_0$

and as a capacity

$$C'_0=C_0+S_a^2/p^2C_0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (48a)$$

in series with a resistance

$$R'_0=S_a/(S_a^2+p^2C_0^2) \quad . \quad . \quad . \quad . \quad . \quad . \quad (48b)$$

when $p<p_0$.

For a system of class B and of one degree of freedom the system behaves as a capacity $\{cC_b/(1-n^2)\}$ in series with a resistance (R_b) if $p<p_0$, and as an inductance $\{L_b(1-1/n^2)\}$ in series with the same resistance if $p>p_0$.

If the system possesses many degrees of freedom, the mode of procedure is to arrange the degrees of freedom in the order of their natural frequencies. If of class A, those below the frequency of the current behave as capacities and those above behave as inductances. By combining these inductances and capacities in series, we obtain the resultant effect as a single inductance or capacity.

A similar process will hold for class B, using parallel combinations for the various degrees of freedom.

The procedure thus outlined only holds when $a_{rs}=0$.

As might be inferred from general considerations the preponderant effect is due to those degrees of freedom whose frequencies lie in the immediate neighbourhood of the source frequency.

The case of a wire in a magnetic field will now be considered in detail.

7. It will be supposed that the damping of the wire is negligible except when the source frequency is in the immediate neighbourhood of one of the natural frequencies of the wire.

If the magnetic field is symmetrical about the centre of the wire then the frequencies of the possible normal modes of vibration are odd multiples of the fundamental frequency which by (40) and (44) is given by

$$p_1^2=\frac{\pi^2}{4l^2}\frac{T_1}{m}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (49)$$

If the source frequency is in the immediate neighbourhood of the fundamental of the wire, all the harmonics of the wire behave as inductances of which the rth has the value

$$L_{rr}/\{1-n^2/(2r-1)^2\},$$

in which $n=p/p_1$.

Further, when the field is uniform throughout the length of the wire, we have by (45), (44) and (40)

$$L_{11}=\frac{64}{\pi^4}\frac{H^2l^3}{T_1}, \quad L_{rr}=L_{11}/(2r-1)^4 \quad . \quad . \quad . \quad . \quad (50)$$

Hence, the apparent inductance due to the wire harmonics is

$$l=L_{11}\sum_{r=2}^{r=\infty}\frac{1}{(2r-1)^4\{1-n^2/(r-1)^2\}}.$$

By expanding each term of this series in ascending powers of n, it may be written

$$l=L_{11}(a_4+a_6n^2+a_8n^4+\ldots.), \quad . \quad . \quad . \quad . \quad (51)$$

where $a_r=3^{-r}+5^{-r}+y^{-r}+\ldots.$

The values of a_r are known (Dale's Tables, p. 92). Thus :—

$$a_4=0{\cdot}01468, \quad a_6=0{\cdot}00145, \quad a_8=0{\cdot}00015, \quad a_{10}=0{\cdot}00002.$$

The series (51) will give 1 per cent. accuracy for l so long as the source frequency is less than 1·4 times the fundamental 10 per cent. up to twice the fundamental. In the latter case l is less than $0{\cdot}02L_{11}$.

If the field, instead of being uniform, tapers off towards the ends of the wire, the effect of the harmonics is less than that estimated above. Hence, for frequencies up to $2p_1$ we may treat the wire as a system of one degree of freedom with a small extra inductance in series with it.

8. An estimate of the inductance L_{11} for the wire may be obtained as follows: Let the wire be of material of density δ and specific resistance ρ. Then, if the cross-section is a, the resistance R is $2\rho l/a$, and the mass per unit length m is $a \times \delta$. Hence, by (49) and (50),

$$\frac{L_{11}p_1^2}{R} = \frac{8H^2}{\pi^2\rho\delta}, \quad \ldots \ldots \ldots \quad (52)$$

a quantity independent of the length and cross-section of the wire.

Thus, for phosphor-bronze, $\rho\delta \doteqdot 10^5$ c.g.s., so that if a wire of this material is tuned to a frequency of 100 per second and is placed in a uniform field of strength 10,000 c.g.s.,

$$L_{11} \doteqdot 2 \text{ millihenries per ohm},$$

while for other frequencies of tuning and other field strengths L_{11} is proportional to H^2/p_1^2.

9. The notation employed in the preceding sections has been used for convenience in developing the theory. For experimental purposes with systems of class A and of one degree of freedom it is preferable to revert to the notation and nomenclature of an earlier Paper.* A vibrating system of this type behaves as a parallel combination of capacity, conductance and inductance. We will denote the elements of this combination by C_v, S_v, and L_v, and refer to them as the *vibration capacity*, *vibration conductance*, and *vibration inductance* respectively. Since their values are independent of the frequency of the current employed, they will be spoken of generally as the *vibration constants*. It has been shown in the Paper quoted how these constants can be measured on an alternating-current bridge, in which a balance is obtained which is independent of the form of the current.

The method is, however, difficult to carry out, and for most purposes it is preferable to make use of the results of section 6 and to treat the moving system as an inductance and series resistance (equations $4y_a$ and $4y_b$) when the source frequency is less than that of the vibrating system; as a capacity and series resistance when the source frequency is greater than that of the vibrating system. Except for large dampings the apparent inductance or capacity is given to a sufficient degree of accuracy by (47) or (48).

* Butterworth, "Proc." Phys. Soc., XXVI., p. 264, 1914.

Hence, if the source frequency is f and the natural frequency is f_0, the apparent inductance (L_0) is given by

$$L_0 = L_v/(1 - n^2), \quad \ldots \quad (53)$$

in which $n = f/f_0$, when $f < f_0$, and the apparent capacity (C_0) by

$$C_0 = C_v(1 - 1/n^2), \quad \ldots \quad (54)$$

when $f > f_0$.

A consideration of the magnitudes of L_0 and C_0 for a given system shows that it is in general simpler to measure L_0, although by using some form of capacity bridge, say, the

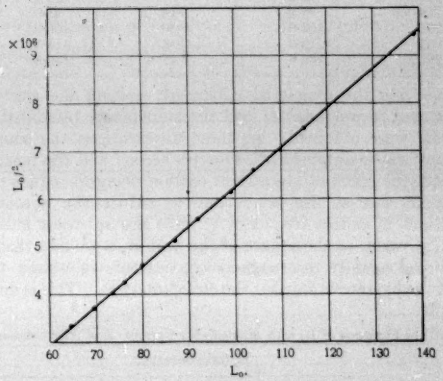


FIG. 1.

modified Carey-Foster bridge, the apparent capacity C_0 could no doubt be measured.

Table I. shows how the apparent inductance of a vibrating wire was found to vary with the frequency of the source. In order to check whether the experimental values satisfy (53) the values of L_0 are plotted against L_0/f^2 in Fig. 1. The result is a straight line, which is in accordance with (53). From the constants of this line,

$$f_0 = 287, \quad L_v = 24{\cdot}3 \text{ microhenries.}$$

Using these values and recalculating L_0 from (53) the values

given in the last column in the table are derived. It is seen that they agree with the observed values to within 2 per cent.

TABLE I.—*Variation of Apparent Inductance L_0 of Vibrating Wire with Source Frequency (f).*

f ~ per sec.	L_0 microhenries.	$L_0 f^2$.	L_0 (calculated).
228	64·5	$3·35 \times 10^6$	65·9
231	69·5	3·71	69·0
234	73·5	4·03	72·5
237	76·0	4·27	76·4
240	80·0	4·61	80·7
243	87·0	5·13	85·5
246	92·0	5·57	91·4
249	99·0	6·14	98·0
252	104·0	6·60	106·0
255	115·0	7·48	114·8
261	139·0	9·47	139·8

By varying the magnetising current exciting the electromagnet used to produce the field the inductances tabulated in Table II. were obtained. In these observations the source frequency was maintained at 240~ per second and the tension increased, the effect of increase of tension being to reduce the inductance, but at the same time to reduce the effect of fluctuations in source frequency. Since the apparent inductance (L_0) varies as the square of the field H, it follows that if we plot $L_0^{\frac{1}{2}}$ against the magnetising current we obtain the form of the hysteresis loop for the electromagnet. This is done in Fig. II.

TABLE II.—*Variation of L_0 with Magnetising Current* (I) *of Electromagnet.*

I Amperes.	L_0 (microhenries).				
	Current direct.		Current reversed.		Current direct Rising.
	Rising.	Falling.	Rising.	Falling.	
0·00	...	1·0	...	1·0	...
0·23	22·4	23·4	16·5	23·0	16·5
0·30	25·6	28·8	22·3	28·4	22·3
0·40	31·8	34·8	30·2	34·4	30·4
0·50	37·4	38·8	35·2	38·8	34·8
0·60	39·6	41·0	38·2	41·6	38·4
0·70	41·6	43·2	41·2	42·8	41·0
0·80	42·6	43·8	...	44·2	42·2
0·90	43·8	44·8	44·2	45·4	43·8
1·00	44·8	46·0	44·8	46·6	45·0
1·40	49·2	...	49·6	...	48·2

10. The results just quoted were obtained for a loop of phosphor-bronze wire of 10 cm. vibrating length and kept

in tension by means of a spring. The arrangement was placed in the air-gap of an electromagnet, the width of the air-gap being 2 mm. and the pole faces 6×4 cm. The measurements were made on a modified Anderson's bridge,* using as detector a Duddell vibration galvanometer. The source was a small alternator, whose fundamental frequency could be varied from 60 to 100∿ per second. The wave-form of the source indicated that it possessed a pronounced third harmonic. As the detector could only be tuned to a minimum

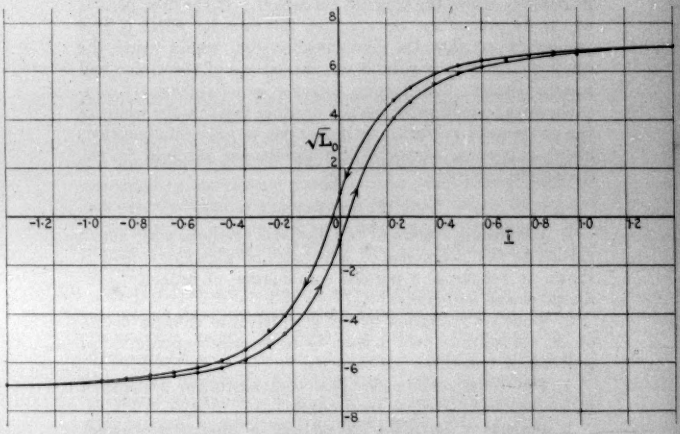


FIG. 2.

frequency of 150∿ per second, the measurments relate not to the fundamental of the source, but to the third harmonic. The fundamental frequency of the source was generally measured by a reed-frequency meter, but as this instrument was not always available use was sometimes made of a Campbell frequency bridge.† The results were concordant. In order to reduce the influence of the fundamental a wave-form sifter,‡ suitable for low frequencies, was employed.

* Butterworth, "Proc." Phys. Soc., XXIV., p. 210, 1912. "Electrician," Vol. LXIX., p. 777, 1912.

† Campbell, "Phil. Mag.," Vol. XV., p. 155, 1908.

‡ Campbell, "Proc." Phys. Soc., XXIV., p. 107, 1912.

11. An interesting application of the theory for systems of class B is supplied by the phenomena of absorption and residual charge in condensers. It has been pointed out * that these phenomena can be imitated electrically by connecting in parallel with a perfect condenser, another condenser (or set of condensers) with a series resistance. Now, the present theory shows that an electrically charged system when set in motion by a varying charge on a condenser, behaves as a set of series combinations of inductance, resistance and capacity arranged in parallel across the original condenser. If the free periods of the dynamical system are high the capacity effect is preponderant, so that the dynamical system would cause the original condenser to exhibit the phenomena of absorption and residual charge. The theory, however, is not sufficiently comprehensive to explain by atomic or other motions the whole of the phenomena exhibited by dielectrics, as only small motions about a position of stable equilibrium are assumed in the dynamical equations.

12. *Summary.*

1. A vibrating system of one degree of freedom when set in motion by the interaction of a current on a magnetic field is shown to behave as a parallel combination of capacity, conductance and inductance.

2. When set in motion by the interaction of charged bodies on an electrostatic field it behaves as a series combination of inductance, resistance and capacity.

3. The corresponding electrical equivalents for systems of many degrees of freedom are obtained.

4. Methods of balancing the systems on alternating-current bridges are discussed, and it is shown how the constants of the systems may be determined by such bridge measurements.

5. Experiments illustrating the theory in the case of a vibrating wire are quoted.

6. It is shown that a moving system of charged bodies is capable of explaining the phenomena of absorption and residual charge in condensers.

In conclusion, I wish to thank Prof. W. W. Haldane Gee for the interest he has taken in the work, and Mr. W. W. Stainer, B.Sc., for assistance in the experimental work.

* Fleming, "Proc." Inst. E.E., Vol. XLIX., p. 323, 1912.

XXXI.—*Numerical Relationships between Electronic and Atomic Constants.*

By H. STANLEY ALLEN, *M.A., D.Sc., Senior Lecturer in Physics at University of London, King's College.*

RECEIVED FEBRUARY 25, 1915.

ONE of the chief aims of the physicist is to reduce the number of quantities in terms of which natural phenomena can be expressed, that is to increase the number of derived units at the expense of the so-called fundamental units. For example, the four units length, mass, time and charge of electricity may be reduced to three by assuming that mass is wholly electromagnetic in origin. The quantum-theory has introduced a new quantity in Planck's constant h. At the Birmingham meeting of the British Association Jeans pointed out that hc, where c is the velocity of light, has the same physical dimensions as the square of an electric charge. "In point of fact, $hc/2\pi$, if not exactly equal, is almost equal to $(4\pi e)^2$, *i.e.*, to the square of the strength of a tube of force binding two electrons. This suggests that the atomicity of h may be associated with the atomicity of e."*

This equivalence between h and e is strongly supported in an interesting Paper by Lewis and Adams†, who conclude that the quantum is merely the square of the fundamental unit of electricity, e, with a simple numerical coefficient depending on the units chosen. They believe that, ultimately, "all universal constants will prove to be pure numbers, involving only integral numbers and π," and decide, without hesitation, "to take the electron charge as the first of the two fundamental constants needed for the final determination of the ultimate units."

Their procedure may be briefly summarised as follows:—

The gas law may be written $PV=nRT$, where n is the number of mols. If m is the total number of molecules and N is the number in one mol, $m=nN$, and the gas law becomes

$$PV=m(R/N)T=mkT,$$

where k is a universal constant, which always occurs to the same

* Jeans, "Report on Radiation and the Quantum-Theory," Physical Society of London, p. 80, 1914.

† Lewis and Adams, "Phys. Rev.," Vol. III., p. 92, 1914.

power as T. The value of k, the gas constant reckoned for one molecule, is Re/F, where F is the Faraday equivalent in the same units as e. This gives $k=1{\cdot}372\times10^{-16}$.

The fundamental assumption made by Lewis and Adams is that the constant a of Stefan's law, $E=aVT^4$, can be expressed in the form

$$a=k^4/(4\pi e)^6.$$

The occurrence of $4\pi e$ here and elsewhere in their results need cause no surprise, as it would be easy to invent a system of units in which $4\pi e$ is the unit of charge.

This assumption gives for the numerical value of a, $7{\cdot}60\times10^{-15}$, from which we may obtain at once as the value of the radiation constant,

$$\sigma=\frac{ac}{4}=5{\cdot}70\times10^{-5},$$

which is in excellent agreement with recent experimental determinations.

Again, by integrating the Planck equation we obtain

$$a=\frac{8\pi^5k^4}{15c^3h^3},$$

and identifying this value of a with that previously assumed we get

$$15c^3h^3=8\pi^5(4\pi e)^6,$$

or

$$ch=\sqrt[3]{\frac{8\pi^5}{15}}(4\pi e)^2.$$

This is the equation giving h in terms of the electron charge. Taking Millikan's value for $e=4{\cdot}774\times10^{-10}$ E.S.U., this gives for h, $6{\cdot}558\times10^{-27}$, which again is in good agreement with the accepted value.

Two Numerical Constants.

The relation between h and e obtained by Lewis and Adams may be written in the form

$$\frac{4\pi^2e^4}{h^2c^2}=\frac{(15/\pi^2)^{\frac{1}{3}}}{(4\pi)^4}=p, \text{ say.}$$

Here p is a pure number whose value is $5{\cdot}30096\times10^{-5}$.

We shall find it convenient later to put $p=q^2$ where

$$\frac{2\pi e^2}{hc}=\frac{(15/\pi^2)^{\frac{1}{3}}}{(4\pi)^2}=q,$$

and q is $7{\cdot}28077\times10^{-3}$.

Electronic Constants.

Now it is a curious fact that several of the fundamental constants of Physics, expressed in the usual electrostatic units, contain simple multiples of the number p.

The charge on the electron, according to the careful determination of Millikan* is

$$\mathbf{e}=4{\cdot}774\pm0{\cdot}009\times10^{-10}.$$

If we put $\mathbf{e}=9p\times10^{-6}$ we obtain

$$\mathbf{e}=4{\cdot}771\times10^{-10},$$

which agrees with Millikan's value within 0·1 per cent.

Bucherer† has reviewed the more recent determinations of $\mathbf{e/m}$.

Classen, 1908	$1{\cdot}776\times10^{7}$ E.M.U.
Bucherer, 1908	$1{\cdot}763\times10^{7}$
Wolz, 1909	$1{\cdot}767\times10^{7}$
Malassez, 1911.....	$1{\cdot}769\times10^{7}$
Bestelmeyer, 1911.....	$1{\cdot}766\times10^{7}$

If we put $\frac{\mathbf{e}}{\mathbf{m}}=p\times10^{22}$ we obtain

$$\frac{\mathbf{e}}{\mathbf{m}}=5{\cdot}30096\times10^{17}\text{ E.S.U.}$$

or $$=1{\cdot}767\times10^{7}\text{ E.M.U.},$$

in close agreement with the most recent determinations. The error is not likely to exceed 0·1 per cent.

The value of $\mathbf{m}$, the mass of the electron, is determined from these two results. Consequently within very close limits we obtain

$$\mathbf{m}=(9p\times10^{-6})\div(p\times10^{22})$$
$$=9\times10^{-28}\text{ gm.}$$

In view of the fact that the fundamental units, the centimetre, the gram and the second are in one sense arbitrary, it is difficult to regard the occurrence of exact powers of 10 in these expressions as other than accidental, just as we regard the approximate expression‡ for the velocity of light, viz., 3×10^{10}, as a coincidence. The repeated coincidences do, however, suggest a question as to whether there may not be some accidental connection between the units of length, mass and

* Millikan, "Phys. Rev.," Vol. II., p. 143, 1913.

† Bucherer, "Ann. d. Physik," Vol. XXXVII., p. 597, 1912.

‡ The accurate value is smaller than the conventional one by about 1 part in 3,000.

time involving the number 10. Such a connection may arise from the way in which the units were originally defined. The metre was intended to be one ten-millionth of the quadrant of the earth from the equator to the pole. The second is defined with reference to the period of axial rotation of the earth. If we express the velocity, v, of a point on the equator in terms of the "natural unit of velocity" c, we find $\frac{v}{c}=\frac{1}{2^3\times 3^4\times 10^3}$. If it were possible to express such a velocity in terms of the mass and dimensions of the earth, a relation between the fundamental units could be established. But as no satisfactory theory determining the rate of rotation of a planet has ever been given, there appears no hope of further progress in this direction.

The only alternative is to consider the possibility of a connection arising from the definition of the gram. The kilogram was intended to have the same mass as that of a cubic decimetre of water at the temperature of its maximum density. Now, if N be Avogadro's constant,* one gram of water contains N/18 atoms of oxygen and N/9 atoms of hydrogen. The atomic volume of hydroxylic oxygen, according to Le Bas, is exactly twice the atomic volume of hydrogen. Consequently if we knew the exact volume to be assigned to the atom of hydrogen (assumed spherical), it should be possible to determine the volume occupied by the given number of atoms when the density is a maximum. Unfortunately we are at present unable to specify the atomic volume of hydrogen, that is its sphere of action, in terms of primary constants. It is, however, suggestive to find a cube root occurring in the numerical constant q. This might be introduced in expressing the diameter of the sphere of action in terms of the volume. We conclude that although it is not possible to determine the form of the connection between the centimetre and the gram from our present knowledge, such a connection must exist, and there is no reason why it should not involve the number 10.

Derived Quantities.

If we accept the expressions for e, m and c in the forms

$$e=9p\times 10^{-6}, \qquad m=9\times 10^{-28}, \qquad c=3\times 10^{10},$$

as being at least arithmetically convenient, it is clear that we

* It is shown later in the Paper that N is approximately $\frac{p}{9}\times 10^{23}$.

can express all derived units depending upon them in terms of simple integers (2, 3 or 4), powers of 10, p and π. As some of these derived quantities contain $\sqrt{p}$ it will be convenient to put $p=q^2$ and use q in our expressions.

The following results may serve as illustrations :—

$$e=(3q\times 10^{-3})^2,$$

$$e/m=(q\times 10^{11})^2,$$

$$m=(3\times 10^{-14})^2,$$

$$h/2\pi=\text{the unit of angular momentum}=(3q)^3\times 10^{-22},$$

$$he/4\pi mc^2=5 \text{ magnetons}=27q^5/2c^2=\frac{3}{2}\left(\frac{q}{10^4}\right)^5,$$

$$a=\text{radius of electron}=\frac{2}{3}\left(\frac{q}{10}\right)^4,$$

$$A_n=\text{radius of H atom on Bohr's theory}=\left(\frac{q}{100}\right)^2,$$

$$\nu_0=\text{fundamental frequency}=\frac{2\pi^2me^4}{h^3}=\frac{3\times 10^{14}}{4\pi q},$$

$$N_0=\text{Rydberg's constant}=\frac{\nu_0}{c}=\frac{10^4}{4\pi q}.$$

The last result is interesting as affording a means of testing the degree of accuracy to be obtained by these approximations. Substituting the value of q we find $N_0=109{,}300$ instead of the value 109,679 found by Curtis, or 109,724, Bohr's value for a massive nucleus. The calculated value is smaller than the observed by less than 0·4 per cent.

Atomic Constants.

If we turn from quantities connected with the electron to quantities connected with the atom, we find results of a similar character, but the numerical agreement is not nearly so good. The fundamental quantity in connection with the atom is Avogadro's constant, N, the number of molecules in one mol. The estimates which have been given of this number vary widely, but there can be no doubt that the most accurate value is that derived from electrical measurements. We take Millikan's value of $e=4{\cdot}774\times 10^{-10}$. "Since the value of the Faraday constant has now been fixed virtually by international agreement (atomic weight of silver 107·88, E.C.E. of silver 0·01118) at 9,650 electromagnetic units, and this is the number

N of molecules in a gram molecule times the elementary electrical charge, we have

$$N \times 4{\cdot}774 \times 10^{-10} = 9{,}650 \times 2{\cdot}9990 \times 10^{10},$$
$$N = 6{\cdot}062 \times 10^{23}."$$

From the value of N we can deduce at once the mass of the atom of hydrogen, m_H, for $m_H = \frac{1}{N}$. This gives $m_H = 1{\cdot}649 \times 10^{-24}$ gm. The value of m/m_H the "atomic mass of an electron," may be determined without assuming a knowledge of the value of e. For we have, taking Bucherer's final value for e/m,

$$\frac{e}{m} = 5{\cdot}307 \times 10^{17},$$

and

$$\frac{e}{m_H} = 9{,}650 \times 2{\cdot}999 \times 10^{10}.$$

Hence

$$\frac{m}{m_H} = 5{\cdot}453 \times 10^{-4}.$$

This suggests taking $\frac{m}{m_H} = 10p$, in which case

$$\frac{m}{m_H} = 5{\cdot}30096 \times 10^{-4},$$

and there is a difference of about 3 per cent. between the observed and the assumed value. On this assumption the mass of the hydrogen atom would be equivalent to 1,886 electrons, we should have

$$m_H = m/10p = 9 \times 10^{-29}/p,$$

and

$$N = \frac{p}{9} \times 10^{29} = 5{\cdot}89 \times 10^{23},$$

which is about 3 per cent. smaller than Millikan's value.*

It would, of course, be possible to express other quantities depending on the mass of the hydrogen atom in the same fashion, but in view of the considerable error involved, no

* A further comparison with experimental determinations is rendered possible by assuming that the magnetic moment of an electron revolving with the unit of angular momentum is equivalent to 5 magnetons. For the revolving electron the magnetic moment $= 27q^5/2c^2 = 3{\cdot}0689 \times 10^{-31}$ E.S.U. $= 92{\cdot}066 \times 10^{-22}$ E.M.U. If we divide Weiss's value for the magnetic moment of the atom gram, 1123·5, by Millikan's value for Avogadro's constant we obtain as the magnetic moment of the magneton $18{\cdot}54 \times 10^{-22}$ E.M.U., and for 5 magnetons $92{\cdot}7 \times 10^{-22}$ E.M.U. The calculated and the experimental values differ by less than 1 per cent.

useful purpose would be served. The results suggest that the relation between the mass of the hydrogen atom and that of the electron is of a more complicated character than the relations between the electron constants.

ABSTRACT.

It has been pointed out by Jeans that hc, where h is Planck's constant and c is the velocity of light, has the same physical dimensions as the square of an electric charge. Lewis and Adams have suggested a relation between these quantities of the form

$$ch = \sqrt[3]{\frac{\pi^7}{15}}(4\pi e)^2.$$

This may be written

$$\frac{2\pi e^2}{hc} = \frac{(15/\pi^2)^{1/3}}{(4\pi)^{1/3}} = q,$$

where q is $7{\cdot}28077 \times 10^{-3}$. The square of this numerical constant is $p = 5{\cdot}30096 \times 10^{-5}$. The charge e on an electron in E.S.U. is found to be, within 0·1 per cent., $9p \times 10^{-6}$. The ratio e/m of the charge to the mass is found to be $p \times 10^{22}$, with the same order of accuracy. The occurrence of powers of 10 in these expressions may be accidental or may depend on the way in which the units of length, mass and time were originally defined. Derived quantities, depending on e, m and c, can be expressed in terms of simple integers (2, 3 or 4), powers of 10, p and π. The "atomic mass of an electron" is approximately $10p$.

DISCUSSION.

Prof. O. W. RICHARDSON said it was curious that the value of e and e/m should happen to be simple multiples of the quantity p. These two quantities were the key to the situation, as the other relations followed more or less directly from them.

Mr. S. D. CHALMERS said it was possible almost with any set of quantities to obtain striking numerical relationships. It was only when a physical significance was sought for that it could be seen whether they were anything but coincidental.

Dr. W. WILSON said that if the units of mass, length, temperature difference, &c., were so chosen as to make the universal constants such as Planck's h, the gravitation constant, &c., each equal to unity, the unit of mass which had to be adopted was p gram.

Prof. ZELENY emphasised the importance of searching for the physical significance attached to these relationships.

XXXII. *On a Method of Calculating the Absorption Coefficients of Different Substances for Homogeneous X-radiation.*

By H. MOORE, *A.R.C.S., B.Sc., Assistant Lecturer in Physics, University of London, King's College.*

RECEIVED APRIL 30, 1915.

IT has been shown as a result of the work of various observers,* that the ionisation produced in a gas when a beam of X-rays passes through it is due to electrons liberated in the gas by the X-rays. If we take two gases at the same pressure, and subject them to identical beams of X-rays, the numbers of electrons liberated in the gases per unit length of the X-ray beam are not identical, the quantity of electronic radiation liberated in a gas depending on the chemical nature of the gas. This electronic radiation is an atomic phenomenon; it depends solely on the nature and numbers of the atoms constituting the gas, being entirely independent of the way in which these atoms are combined.†

In a recent Paper ("Proc." Roy. Soc., May, 1915) it was shown by the author that the quantity of electronic radiation liberated from an atom by a beam of X-radiation is directly proportional to the fourth-power of the weight of the atom—*i.e.*, if two elemental gases, having equal numbers of atoms per unit volume, are subjected to identical beams of X-rays, the numbers of electrons liberated in these gases per unit length of X-ray beam are proportional to the fourth powers of their atomic weights. The liberation of electrons in a gas by the passage of an X-ray beam through it is accompanied by an absorption of the X-rays in the gas. The absorption of the X-ray beam in a given length of a gas is proportional to the number of electrons liberated by the X-rays in traversing this portion of its path. This being so, it is evident that the coefficients of absorption of a given type of X-rays in two elemental gases must be proportional to the fourth power of their atomic weights, provided the number of atoms per unit volume is the same in the two gases.

* Barkla and Simons, "Phil. Mag.," Feb., 1912; C. T. R. Wilson, Roy. Soc. "Proc.," June, 1912; Barkla and Philpot, "Phil. Mag.," June, 1913.

† H. Moore, "Phil. Mag.," Jan., 1914.

The liberation of electrons in a compound gas by X-rays being an atomic phenomenon, the absorption of X-rays in such a gas must also be atomic—*i.e.*, the absorption in a compound is the sum of the absorptions that would be produced, if the various constituent atoms were present in the same numbers but not in combination. This atomic property of the absorption in compounds was shown by actual experiment by Benoist as early as 1901.

The present Paper shows how, for any type of homogeneous X-radiation, the absorption coefficient in a hypothetical, monatomic vapour of any element at a pressure of 76 cm. can be calculated, if the coefficient of absorption of these rays is known in air, and provided that the radiations excited in the hypothetical vapour belong to the same sets of series as those excited in air by the same incident beam (*see* pp. 436 and 437). From the absorption coefficient calculated for this hypothetical vapour the absorption coefficient for the same type of X-rays can be calculated for the same element in any condition, a simple density law holding good whether the element is solid, liquid, or gaseous. By an additive law, the absorption coefficient of any compound can be calculated from the absorptions of the various constituent elements, and thus it should be possible to calculate eventually the coefficient of absorption of any substance of known constitution for any type of X-radiation whose absorption coefficient is known in air, provided the radiations excited in all these cases belong to the same sets of series (*see* pp. 436 and 437). It is shown in the Paper to which reference has been made* that if a monatomic vapour of an element at a pressure of 76 cm. could be subjected to a homogeneous beam of X-rays from copper, the number of electrons liberated in unit length of the vapour could be expressed in the form $\frac{(\text{atomic weight})^4}{\text{a constant}}$. If the number of electrons liberated by the same beam of X-rays in passing through unit length of air is taken as unity, the mean value of this constant was found to be $1{\cdot}05 \times 10^5$. This constant is based on the *ratio* between the electronic radiations liberated in the vapour and in air, and should, therefore, have the same value for all types of X-radiation, provided the result of the absorption is similar in all cases (*see also* pp. 436 and 437). Calculations based on this assumption show a fairly good agreement with observed values for radiations

* Moore, "Proc." Roy. Soc., May, 1915.

other than copper, thus bearing out this conclusion. It has, of course, been abundantly shown that the absorption coefficients of two materials bear a constant ratio for all X-radiations, provided the types of secondary radiation excited in the two materials are similar.

A beam of X-radiation passing through unit length of one of these hypothetical vapours would cause the liberation of a quantity of electrons which can be calculated in terms of the quantity emitted in unit length of the same X-ray beam in air, the aperture being supposed to be the same in both cases. The absorption coefficients for this hypothetical vapour and for air would bear the same ratio to each other as these quantities of electronic radiation, and thus, if the absorption coefficient in air is known for this X-ray beam, the absorption coefficient in the vapour can be calculated. The density of the vapour is calculated from its atomic weight.

Absorption in Aluminium.

Aluminium is taken as an example of the method of calculation, because the absorption coefficients of different radiation are usually determined in thicknesses of aluminium.

A monatomic vapour of aluminium at 76 cm. would have an absorption coefficient for any type of radiation, equal to $\frac{(27)^4}{105{,}000}$ times that of air for the same radiation—*i.e.*, for monatomic aluminium vapour at 76 cm.

$$\lambda = 5{\cdot}06 \times \lambda_{\text{air}}.$$

The absorption coefficients of this vapour for different types of radiation have been calculated on this assumption, and will be found in column 3, Table I. These numbers are based on the absorption coefficients found experimentally for the same radiation in air.

The density of the vapour would be 0·00121 grm. per cubic centimetre, and the density of solid aluminium is 2·7 grm. per cubic centimetre; applying the density law, the actual coefficient of absorption in aluminium sheet will be 2,230 times the absorption coefficient for this hypothetical vapour, or

$$\lambda_{Al} = 2{,}230 \times 5{\cdot}06 \times \lambda_{\text{air}}.$$

Values for λ_{Al}, calculated for different radiations by this method, are given in column 4, Table I., and in column 5 of the same table are the coefficients actually determined by various experimenters.

TABLE I.—*Absorptions in Aluminium.*

Radiator.	λ in air (observed).	λ in hypothetical aluminium vapour (calculated).	λ in aluminium sheet (calculated).	λ in aluminium sheet (observed).
Iron	0·0201	0·1017	227	239
Copper	0·0109	0·055	123	128
Zinc	0·00898	0·0455	102	106
Bromine	0·00389	0·0197	43·9	44·0
Silver.....	0·00076(2)	0·0038(5)	8·5	6·75
Tin.....	0·00050(3)	0·0025(4)	5·6	4·24

NOTE.—The observed values are taken or deduced from the values given in Kaye's book on "X-Rays."

The agreement between the observed and calculated values is quite close for the iron, copper, zinc and bromine radiations, but is less exact for the harder radiations of silver and tin. Even in these cases, the difference is not more than 25 per cent., and is probably within the limits of accuracy obtainable in the majority of the experiments from which the observed values are quoted, at any rate for the harder rays. The determination of absorption coefficients is rendered extremely difficult by the enormous scattering correction which has to be applied, this correction being greater for the harder rays than for rays of longer wave-length. The scattering coefficient is only approximately known, and this could easily lead to errors of 25 per cent. or over, in the values obtained for the absorption coefficients of the harder rays.

Absorptions in Other Elements.

If different radiations are supposed to be absorbed in various elemental substances, and values for the coefficients of absorption are calculated as for aluminium, a sudden disagreement occurs in some cases between the calculated and observed values as the atomic weight of the absorber becomes greater than that of the radiator.

This is shown in the following table :—

TABLE II.—*Absorptions in Various Metals.*

Absorber.	λ (calculated) for radiations from					λ (observed) for radiations from				
	Fe.	Cu.	Br.	Ag.	Sn.	Fe.	Cu.	Br.	Ag.	Sn.
Mg	107	57·3	20·6	4·03	2·67	139	72	...	3·82	...
Al.....	227	123	43·9	8·5	5·6	239	128	44	6·75	4·24
Fe	630	330	112	21·9	14·5	520	21,000	...	137	...
Ni	765	413	148	29·0	19·1	746	458	...	22	...
Cu	986	531	191	37·5	24·7	850	552	...	21·7	...
Ag	5,600	3,300	1,085	213·5	141	4,000	225	...	139	...

The values for magnesium and aluminium give approximate agreement, wherever observed values are available for comparison with the calculated values. The calculated and observed absorptions in iron, however, though of the same order of magnitude for iron radiation, are entirely different for the radiations from metals of higher atomic weight, the calculated values being much greater than the observed.

Nickel and copper, on the other hand, show an approximate agreement throughout the whole range of radiations considered, notwithstanding the fact that the atomic weights of the radiators are some above and some below the atomic weights of the absorbers.

Silver is different again, giving calculated and observed values of the same order of magnitude for radiation from iron, but the values for all the other radiations are of entirely different orders.

These anomalies are connected with the selective absorption of X-radiation in the different metals, and are probably due to the fact that in some cases the homogeneous radiation corresponding with the K series is absent (*see also* p. 437).

Absorption in Compounds.

With compounds, the absorption coefficients calculated on the additive law by similar methods agree fairly well with the observed values where known, provided the constituent elements have atomic weights below that of the radiator.

As an example of this, the value was calculated for the absorption coefficients of SO_2 for different radiations. The coefficients for monatomic vapours of sulphur and oxygen at 76 cm. were first calculated and the value for SO_2 determined by addition; the results obtained are given in Table III. A similar set of values calculated for ethyl bromide gave no agreement whatever with the observed values, bromine having an atomic weight greater than that of the radiator used, the K radiation from bromine being, therefore, not excited (*see* p. 437).

TABLE III.—*Absorptions in* SO_2.

Absorber.	λ for radiations from				
	Fe.	Cu.	Br.	Ag.	Sn.
Monatomic sulphur vapour at 76 cm. (hypothetical).	0·201	0·1085	0·0389	0·00762	0·00503
Monatomic oxygen at 76cm. (hypothetical)	0·0125	0·00672	0·00241	0·00047	0·00031
SO_2 (calculated from above)	0·226	0·122	0·0437	0·0085	0·00565
SO_2 (observed)*	0·24	0·134	0·050	0·0079	...

* Barkla and Collier, "Phil. Mag.," June, 1912.

The sudden disagreement between the calculated and observed values of the absorption coefficients, which occurs as the atomic weight of the absorber becomes greater than that of the radiator, is very marked in the case of iron, though it is hardly shown at all with copper and nickel. It would appear that this discontinuity is associated with selective absorption.

If a radiation of definite wave-length, say, the K radiation from some element, is allowed to fall on materials of atomic weight less than that of the radiator, the absorption would produce characteristic radiations from the absorber belonging to all the series K, L, M, &c. When the atomic weight of the absorber is greater than that of the radiator, the K radiation from the absorber is no longer excited, and the absorption produces the L, M, &c., radiations only. The sudden cessation of the K radiation might be expected to produce a discontinuity in the relation between the absorption coefficient and the atomic weight of the absorber, and this would correspond with the change noticed.

It might be supposed that the atomic absorption for a given radiation is an additive quantity.

For the K radiation excited in the absorber, the absorption may be some quantity Ax^4, where A is a constant, and x the atomic weight of the absorber. The L radiation excited demands a further absorption Bx^4, the M and other radiations (if any) requiring absorptions Cx^4, Dx^4, &c. In each case a fourth-power law holds good, but a discontinuity occurs when one of the radiations ceases to be excited, the absorption coefficients being given by $(A+B+C+D+..)x^4$ when all the radiations are given off, and changing to $(B+C+D+..)x^4$ when the K radiation ceases.

Prof. Bragg and Mr. Pierce * have shown that the atomic absorption coefficient does vary in some such way. A fourth-power law is obeyed fairly accurately up to a certain value of the atomic weight, but at this point a discontinuity occurs. For absorbers of higher atomic weight the atomic absorption is still proportional to the fourth power of the atomic weight or of the atomic number, but the constant coefficient is different.

In their calculations, the atomic absorptions were compared with the fourth power of the atomic numbers of the absorbers. The accuracy of the absorption coefficients determined is not,

* "Phil. Mag.," Oct., 1914.

however, sufficiently great to determine whether the atomic number or the atomic weight should be used.

There is no doubt that the absorption of X-radiation requires much further investigation before it is at all completely understood, and the present Paper is only offered as a possible solution of one part of the phenomena connected therewith. In any futher investigations it will be essential to ensure strict homogeneity of the X-ray beam used, and every attempt should be made to obtain a higher order of accuracy than has been attained in most of the experiments up to the present.

ABSTRACT.

The action of X-radiation when passing through a gas is to liberate electrons from the gas. The number of electrons emitted by any atom in a beam of X-rays is proportional to the fourth-power of its atomic weight (or possibly its atomic number). (Moore, "Proc." Roy. Soc., May, 1915.) Thus, equal numbers of atoms of different elements, when subjected to similar X-ray beams, will liberate amounts of electronic radiation proportional to the fourth powers of the atomic weights of the elements.

The absorption coefficients are proportional to the amounts of electronic radiation liberated, and, therefore, the absorptions of two elements, when equal numbers of atoms are present, will be proportional to the fourth powers of their atomic weights.

The corpuscular radiation liberated in the vapour of an element if it could be obtained as a monatomic vapour at 76 cm. can be expressed as $1{\cdot}05 \times 10^5 \times (\text{atomic weight})^4$, taking the corpuscular radiation in air as unity. The absorption coefficient of such a vapour would, therefore, be this number of times the coefficient of absorption of air for the same type of X-radiation. The absorption of any element is proportional to the number of atoms present, and having calculated the absorption in a hypothetical vapour of this type, the absorption in the same element in any condition can be calculated by a simple density law.

This is done in the Paper for several elements (metals), and also, assuming an additive law, it has been calculated for some compounds. The agreement between the calculated values and the values obtained by different observers by direct experiment is quite close over a considerable range of radiations and absorbers. When, however, the atomic weight of the absorber is higher than that of the radiator, so that the K series is absent from any secondary radiation excited in the absorber, the agreement ceases. A possible explanation of this is suggested in the Paper.

DISCUSSION.

Prof. O. W. RICHARDSON thought the Paper made a considerable addition to our knowledge of the process of ionisation by X-rays. The primary rays in being absorbed produce some ionisation, and, in addition, excite secondary radiation, which in turn produces more ions. The Paper helps to make clear the part played by these two causes in the production of the total ionisation.

Dr. S. Russ felt that the Paper conveyed the impression that one would find in practice the same value for the absorption coefficient as obtained by calculation. The phenomenon of selective absorption must surely set severe limits to the range of wave-length over which these relations hold. This range should be specified. How would the author calculate the absorption coefficients, say, for aluminium and water? If there were no selective absorption λ/ρ should be constant, but he had recently found it to vary over a wide range.

Prof. Nicholson said the results were of interest to those engaged in the attempt to construct models of atoms and molecules. The phenomena appeared to be atomic in nature, thus bearing out the idea brought forward by Bragg that the atom preserves its identity in the molecule. In connection with the law connecting the absorption with the fourth power of the atomic weight, it would be interesting to know whether it was really the atomic weight or the atomic number that should be employed.

The Author replied as follows: The formula should only be used where all the types of radiation excited in air are also excited in the material. He had had considerable trouble in getting reliable coefficients owing to the large and uncertain scattering corrections which had to be applied. In one case he believed an observed value of 0·42 had to be corrected to 0·2 or less. He had calculated some coefficients for gases, and with SO_2 the agreement was as good as for solids. It should be possible to calculate the true absorption coefficient for water and aluminium, but he could not say how near these would be to values actually determined on account of the uncertain scattering. He had experiments in progress which should determine whether the atomic weight or the atomic number was the significant quantity.

At the meeting of the Society held on Friday, May 28, 1915, at University of London, King's College, the following demonstrations and exhibitions were given :—

Two Experiments Illustrating Novel Properties of the Electron Currents from Hot Metals were shown by Prof. O. W. RICHARDSON, *M.A., D.Sc., F.R.S.*

THE first experiment demonstrates the cooling of a tungsten filament when an electron current is allowed to flow from its surface. This effect is analogous to the cooling due to latent heat when a liquid evaporates, or to the similar phenomenon due to the heat of reaction when a gas is emitted by chemical decomposition of a solid.

An experimental lamp containing a fine filament of double tungsten is placed in one arm of a balanced Wheatstones bridge actuated by the current which heats the wire. When the electron current is allowed to flow, by completing a side circuit from an electrode inside the lamp to a point in the adjacent arm of the bridge, the galvanometer is deflected in a direction which corresponds to a reduction of the resistance (and temperature) of the hot filament. The precautions necessary fully to eliminate disturbances due to various subsidiary phenomena are considered in the following Papers by H. L. Cooke and the writer ("Phil. Mag.," Vol. XXV., p. 624, 1913, and *ibid.* Vol. XXVI., p. 472, 1913).

The second experiment, in which a similar experimental lamp is used, demonstrates the flow of electron currents from a hot filament to a surrounding cylinder *against* various opposing potential differences up to about 1 volt. On account of the large currents from tungsten this effect can easily be shown on a galvanometer. The data can be used to find the velocities of the emitted electrons (cf. "Phil. Mag.," Vol. XVI., p. 353; Vol. XVII., p. 890; Vol. XVIII., p. 681).

On High Permeability in Iron.

PROF. E. WILSON gave a short account of the experiments, and exhibited the apparatus, which he had recently described in Papers read before the Royal Society. In these was dis-

cussed the increase of permeability of silicon iron produced by heat treatment with application of magnetic fields during cooling, and by demagnetisation while screened from the earth's magnetic field.

An Experiment Showing the Difference in Width of the Spectrum Lines of Neon and Hydrogen was shown by Mr. T. R. MERTON.

FOR accurate measurements of the widths of spectrum lines an interference method must be employed, in which the optical difference of path between the inferfering beams can be varied, the widths being calculated from the limiting difference of path at which interference fringes can be seen. By "crossing" a Fabry & Perot etalon with a single prism spectroscope it is possible to discriminate between lines arising from different elements, by the "visibility of the fringes." In the experiment shown a vacuum tube containing neon and hydrogen is examined in this way. The neon lines being narrow, show sharp interference fringes, but for the hydrogen lines, which are broader, the limiting order at which interference can be seen is too low for fringes to be visible.

XXXIII. *The Cohesion of Solids.* *By* Prof. Herbert Chatley, *D.Sc.* (*Lond.*), *M.I.C.E.I.*

Received December 13, 1914.

It is a remarkable fact that the force which links the particles of the earth to those of the distant sun is much better understood than that which connects those earth particles together. The most plausible hypothesis would seem to be that which identifies the two. Lord Kelvin favoured this view, but there are several difficulties.

Two salient facts arise from any investigation, viz. :—

(*a*) The attractive force which we term cohesion has a very minute range, but appears to be considerably greater than gravitation at minute distances.

(*b*) There is a repulsion between small particles of matter as well as an attraction.

In order to appreciate the second fact correctly it is necessary to make a preliminary conjecture as to the nature of the ultimate particles of matter. Elaborate investigation has shown that solid matter has a whole series of discontinuities—

1. Crystalline structure bounded by interfaces, with occasional minute cavities between the crystals. This structure is usually visible in microscopic sections. In the case of "amorphous" solids it may, perhaps, be assumed that the crystals are of molecular dimensions.

2. Molecules consisting of a geometrically arranged group of two or more atoms bound by electrostatic (?) forces (chemical affinity) immensely superior in intensity to gravitation or cohesion.

3. Atoms consisting of electrons, each atom forming a dynamic complex of immense stability.

4. Possible etheric sub-structure, such as Osborne Reynolds' "granules" of "negative matter."

It would appear that under ordinary circumstances the atoms behave as indefinitely rigid minute solids, so that discontinuity of substance or delay in the transmission of momentum arises solely from the fact that the molecules are separated by appreciable distances.

The elastic compressibility of solids shows that—

(*a*) The molecules or the atoms in the molecules may be brought closer together.

(*b*) Such closer proximity develops a mutual repulsion in the molecules, and "strain energy" is stored in the substance,

which will return the parts to their original positions (or nearly so) when the compressing force is removed.

Elastic tension similarly indicates the presence of a mutual attraction and separability.

A solid under no stress has its particles in a state of equilibrium. Disturbance of that state in the direction of separation or approximation is accompanied by a restoring force. Temporary application of stress is, therefore, followed after its removal by harmonic oscillations, which are at first congruous and then become degraded into heat, which is radiated or conducted away.

The application or removal of heat causes a difference in the equilibrium position of the particles. The addition of heat causes expansion, implying an increase in the mutual repulsion, diminution in the attraction, or both. Conversely, the subtraction of heat implies a decrease in the repulsion, increase in the attraction, or both. This phenomenon, together with certain others, has led physicists to believe that heat in solids consists in a vibration of the particles, the kinetic energy varying with the total heat. If the absolute zero of temperature is also the zero of heat, the mean kinetic energy of the particles varies approximately as the absolute temperature.

If it is assumed that the vibrations are linear, simple, harmonic oscillations (an assumption which is certainly not quite true), then we may write approximately :—

Total heat = number of molecules × mean kinetic energy per molecule.

$$\mathrm{JM}s\theta = n^3 m \cdot \frac{\pi^2 \cdot \Delta^2}{\mathrm{T}^2},$$

where M = mass of unit volume $= n^3 m$,

s = mean specific heat from zero absolute to θ,

θ = temperature (absolute), centigrade,

J = mechanical equivalent of heat, ergs per gram-calorie,

n^3 = molecules in unit volume,

m = mass of a molecule,

Δ = mean amplitude of oscillation,

T = mean periodic time of oscillation.

$$\frac{\Delta}{\mathrm{T}} = \frac{1}{\pi}\sqrt{\mathrm{J}s\theta}.$$

The mean velocity of the particles is $\frac{4\Delta}{\mathrm{T}}$.

For example, steel at temperature of 300 deg. absolute (27°C., 80°F.),

$$\frac{\Delta}{T}=\frac{1}{\pi}\sqrt{4{\cdot}2\times10^{7}\times0{\cdot}11\times300}=11{,}850 \text{ cm. per second.}$$

$$v=\frac{4\Delta}{T}=47{,}400 \text{ cms./sec.}=1{,}555 \text{ ft./sec.}$$

A dull red heat corresponding to a temperature of about 600°C. (say 900 deg. abs.) will then involve a velocity of about 1·7 times the above, say 2,500 ft./sec., or 80,000 cms./sec. Optics shows that the vibrations in light of that colour have a frequency of about 400×10^{12} per second, so that on these assumptions

$$\Delta \text{ at 900 deg. abs.}=\frac{80{,}000}{400\times10^{12}\times4}=0{\cdot}5\times10^{-8}.$$

2Δ, the molecular swing, would then be $1{\cdot}0\times10^{-8}$ cm. which agrees moderately well with otherwise determined molecular dimensions.

Such oscillators would present great resistance to compression applied in the direction of oscillation, but, per se, their attraction would not be different from the usual gravitational one, and even at the small distance of $0{\cdot}5\times10^{-7}$ cm. the gravitational force is small.

The force of gravitation follows the law

$$t \text{ (dynes)}=\frac{Gm_1m_2}{d^2},$$

where $G=6{\cdot}6\times10^{-8}$; m_1, m_2 are the attracting masses (grammes); d is the distance, centre to centre, centimetres.

If we write $m_1=m_2=\frac{M}{n^3}$, where $n=\frac{1}{d}$,

$$t=G\frac{M^2}{n^4}=GM^2d^4,$$

where M=mass per unit volume=specific gravity.

Thus, in steel, specific gravity, say 8·0,

$$t=6{\cdot}6\times10^{-8}\times64\times(0{\cdot}5\times10^{-7})^4,$$

an excessively minute force, apparently quite incapable of explaining cohesion.

It seems difficult to understand how the molecules can be brought much closer together than the above-mentioned distance without developing repulsive forces greatly in excess of t.

At very low temperatures there is a contraction which may reduce the distance to perhaps half that given. Cohesion does increase at such low temperatures, but still the gravitational force seems insufficient.

Kelvin was of opinion that cohesion could be explained by supposing the actual mass of substance to be concentrated into volumes quite small as compared with the whole volume. The density at such centres would be extremely high, and this intense density would, he thought, so increase the force of gravity between pairs of molecules very close together as to account for cohesion without necessitating any deviation from the inverse square law ("Pop. Lect.," Vol. I., and "Proc." Roy. Soc. Edin., April 21, 1862, Vol. IV.).

It appears to me that an inconsistency is involved in this hypothesis. In a mass of apparently homogeneous substance the average distance from centre to centre of the molecule pairs will not have any directional relation, so that in the case of a longitudinal tension resisted by cohesion (say in a bar under tension) there seems no particular reason to suppose that the longitudinally cohering pairs are closer together than the lateral pairs, and yet Kelvin's hypothesis would seem to require this.

NOTE.—Sir James Dewar, in the "Encyclop. Britt.," 11th edition Vol. XVI., article "Liquid Gases," p. 756c, says :—

"COHESION.—The physical force known as cohesion is greatly increased by low temperatures. This fact is of much interest in connection with two conflicting theories of matter. Lord Kelvin's view was that the forces that hold together the ultimate particles of bodies may be accounted for without assuming any other forces than that of gravitation, or any other law than the Newtonian. An opposite view is that the phenomena of cohesion, chemical union, &c., or the general phenomena of the aggregation of molecules, depend on the molecular vibrations as a physical cause (Tolver Preston, "Physics of the Ether," p. 64). Hence, at the zero of absolute temperature, this vibrating energy being in complete abeyance, the phenomena of cohesion should cease to exist and matter generally be reduced to an incoherent heap of "cosmic dust." This second view receives no support from experiment. Atmospheric air, for instance, frozen at the temperature of liquid hydrogen, is a hard solid, the strength of which gives no hint that with a further cooling of some 20 deg. it would crumble into powder. On the contrary, the lower the scale of temperature is descended the more powerful become the forces which hold together the particles of matter."

Consider the case of a square bar 1 cm. on a side, subject to a critical tension f=2,000,000 grammes. Let the specific gravity be 8, and the number of molecules per centimetre 2×10^8.

This gives very approximately the conditions in an iron or steel bar somewhat above the elastic limit.

If we assume that the cohesion is fairly well represented by $(2\times10^8)^2$ inter-molecular links, the tension in each link is $(2\times10^6)/(4\times10^{16})=0{\cdot}5\times10^{-10}$ grammes, or say 5×10^{-8} dynes. This is, then, the tension between a pair of molecules on the two sides of an imaginary normal section. Assuming that the bond along the line connecting the two is (almost) wholly due to their mutual attractions, we have

$$5\times10^{-8}=\frac{Gm^2}{d^2},$$

where G is the gravitational constant, m is the mass of a molecule and d is the distance from centre to centre of a linked pair. G being $6{\cdot}6\times10^{-8}$, and our figures being very rough, we have approximately

$$m^2/d^2=1, \text{ so that } m=d,$$

$$m(\text{grammes})=8/(2\times10^8)^3=10^{-24} \text{ grammes, and } d=10^{-24} \text{ cm.}$$

NOTE.—m here agrees with the mass of a hydrogen atom rather than an iron one, but this discrepancy is immaterial for our purpose.

This makes d smaller than the reputed diameter of an electron ($3{\cdot}0\times10^{-13}$ cm.), or even than a Reynolds' granule ($5{\cdot}5\times10^{-18}$ cm.), which is apparently absurd.

If the number of molecules is reduced to as low as 2×10^6 per centimetre, we have

$$\frac{2\times10^6}{4\times10^{12}}=0{\cdot}5\times10^{-6} \text{ grammes, or } 5{\cdot}0\times10^{-4} \text{ dynes.}$$

$$5{\cdot}0\times10^{-4}=\frac{6{\cdot}6\times10^{-8}\times\left(\frac{8{\cdot}0}{8\times10^{18}}\right)^2}{d^2}$$

and d=approximately 10^{-20} cm.

Even allowing for the additive effect of other molecules than those in the pair on the tension along the line between the pair it seems difficult to increase d to a reasonable figure, and in any case there seems to be no very good reason why it should differ from the reciprocal of n.

If, on the other hand, we assume that cohesion follows a law

$$t=\frac{Gm^2}{d^\gamma}$$

and d *is* equal to $1/n$, so that $t=\frac{Gm^2}{\left(\frac{1}{n}\right)^\gamma}$,

where t is the link between two molecules, m is the mass of a molecule, d is the inter-molecular distance, and γ is an exponent we have

$$\frac{981f}{n^2}=\frac{6{\cdot}6\times10^{-8}\times\left(\frac{8{\cdot}0}{n^3}\right)^2}{\left(\frac{1}{n}\right)^\gamma}$$

approx. $$n^4n^{-\gamma}=2\times10^{-14}$$

say $$n=10^7$$

$$10^{-7\gamma}=2\times10^{-42}.$$

Neglecting the 2, $\gamma=6$, which agrees with Boscovich's theory that the attraction at molecular distances follows a variation with a high exponent.

NOTE.—Relation of inter-molecular distance to size of molecular fields according to the Newtonian law :—

$$\frac{981f}{n^2}=\frac{G\left(\frac{\sigma}{n^3}\right)^2}{d^2}\ ;\ d=\sqrt{\frac{G\sigma^4}{981fn^4}}\quad[\sigma=\text{specific gravity}],$$

so that, other things being the same, $d\propto\frac{1}{n^2}$.

Relation of variational index if inter-molecular distance equals size of molecular field :—

$$\frac{981f}{n^2}=\frac{G\left(\frac{\sigma}{n^3}\right)^2}{\left(\frac{1}{n}\right)^\gamma}$$

$$\gamma\log n=\log\left[\frac{981fn^4}{G\sigma^2}\right]$$

$$\gamma=4+\frac{\log\frac{981f}{G\sigma^2}}{\log n}$$

The above theory assumes that when the molecules are near

the limit of equilibrium the attraction (repulsion being disregarded) follows either the law

$$Gd^{-2} \text{ or } Gd^{-\gamma}.$$

The previous reasoning assumes that for the given stress between the molecules the repulsion is zero. This may not be very incorrect if the strain is such that the molecules have been dragged to the extremity of the molecular field. This condition would be indicated by an extension of not less than one-fiftieth, or in the case of simple tension, where the lateral approach of the molecules reinforces the longitudinal attractions, as much as one-quarter (for steel or iron).

Being ignorant of the arrangement of the molecules and of the laws of their repulsion and attraction, it is almost impossible to say exactly what ratio there is between the actual bond along a line between two molecules and that which there would be if these two were isolated. The ratio cannot well be less than unity, and may, perhaps, be as high as 50. Taking it as unity, let us consider two isolated molecules.

There is a repulsion between them which varies in some inverse sense with the distance from centre to centre. There is also an attraction which similarly varies, but in the position of equilibrium the two are balanced. If brought closer repulsion prevails to an extent which increases continuously with proximity. It may not become infinite, since chemical (*i.e.*, electrical) change may occur such that the atoms rearrange themselves (possibly even interpenetrate), but still closer proximity than this will experience enormous resistance from the dynamic repulsion of the extremely energetic electron systems.

A law of the form

$$t_1 = k_1 d^{-\gamma_1}$$

may serve for the repulsion where t_1 is the repulsion (negative attraction), k_1 is a coefficient, d is the inter-molecular distance, γ_1 is an index not less than unity.

The effective attraction during separation does not increase in this way, but, on the contrary, approaches a limit generally within a distance less than $2d_0$, where d_0 is the position of equilibrium. Outside this range it diminishes and becomes equal to the gravitational attraction at a distance of a few times d_0. (Various experimenters have found that cohesion is inappreciable at distances more than about 10^{-5} cm.)

If we consider the effective bond to be the resultant of an attraction and a repulsion we have

$$t=t_2-t_1,$$

and it will be found that t_2 (the molecular attraction) may also be written in a form

$$k_2 d^{-\gamma_2}$$

and agree with the experimental facts (*see* sketch).

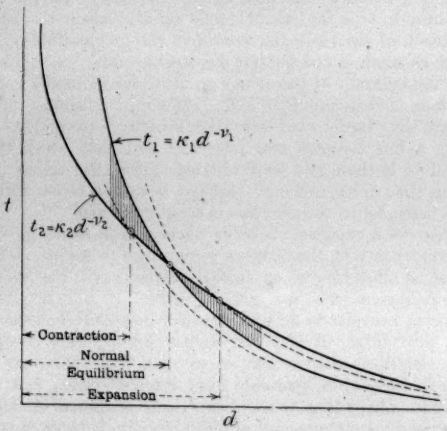


FIG. 1.

Hooke's law shows that for small increases or decreases x in the neighbourhood of d_0, t is of the form cx, where c is a constant. The conditions will be approximately satisfied if the curve representing t_1 is asymptotic to the d axis more rapidly than that representing t_2 and is asymptotic to the ordinate axis (or perhaps to one parallel thereto) less rapidly than that of t_2.

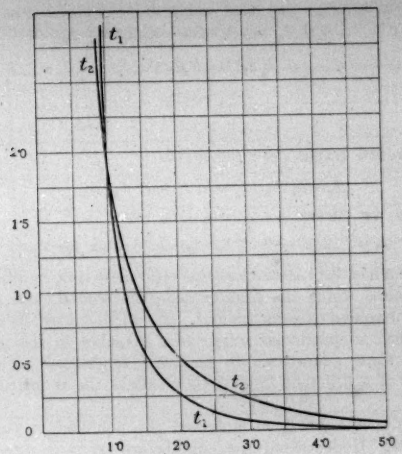
Since at equilibrium $t_1=t_2$ and d is common,

$$k_1 d_0^{-\gamma_1}=k_2 d_0^{-\gamma_2},$$

and since the slope of the t_1 curve exceeds that of the t_2 curve,

$$-\gamma_1 k_1 d^{-(\gamma_1+1)} > -\gamma_2 k_2 d^{-(\gamma_2+1)}$$

and since $$d^{-(\gamma+1)}=d^{-\gamma}\div d$$

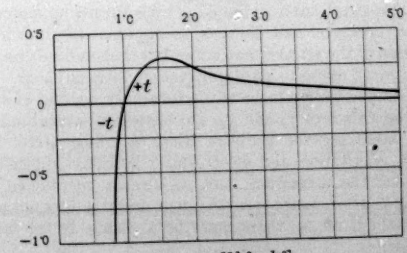


$t_1 = 2d^{-3}$.

$t_2 = 2d^{-2}$.

$\frac{dt_1}{dd} = -6d^{-4}$

$\frac{dt_2}{dd} = -4d^{-3}$



$t = t_2 - t_1 = 2[d^{-2} - d^{-3}]$.

$\frac{dt}{dd} = -2[2d^{-3} - 3d^{-4}]$.

t is a max. when $d = 1{\cdot}5$.

FIG. 2.

for the value d_0,

$$-\gamma_1(k_1 d_0^{-\gamma_1} \div d_0) > -\gamma_2(k_2 d_0^{-\gamma_2} \div d_0)$$

and $$k_1 d_0^{-\gamma_1} = k_2 d_0^{-\gamma_2}.$$

Then $$-\gamma_1 > -\gamma_2.$$

$$\therefore \quad \gamma_1 > \gamma_2.$$

The attached graphs show the curves

$$t_1 = 2d^{-2}; \quad t_2 = 2d^{-2} \text{ and } t_2 - t_1,$$

and also the curves

$$t_1 = 0{\cdot}125\,(d - 0{\cdot}5)^{-4}; \quad t_2 = 2d^{-2} \text{ and } t_2 - t_1,$$

indicating how for functions of this type there is an equilibrium value below which the first exceeds the second, and above which the second exceeds the first. It will, of course, be understood that no particular importance attaches to the indices used in these examples. In fact, the functions themselves may be of a different kind if they satisfy the required conditions.

If the coefficient k_1 is increased the equilibrium point is advanced. If decreased the equilibrium point is retracted. This corresponds to the effect of heat in causing expansion and contraction. Similarly, by sufficiently increasing k_1, the equilibrium point may be abolished, corresponding to the gaseous state.

The resultant curve in the neighbourhood of d_0 corresponds to the stress-strain curve obtained in experiments on materials. On the compression side there is an apparent discrepancy in the increase of the stress-strain ratio, but it should be observed that this curve refers to pure contraction without shear.

There does not seem to be any satisfactory method of ascertaining the values of γ_1 and γ_2, but there are several indirect ways of attacking the problem which are suggestive. The theory of gravitation and electrostatic potential suggest the index 2 for the attraction, but, as already mentioned, the coefficient "G" does not seem to agree, and if it does not agree at infinitesimal ranges there must be a change in the law at some point.

The kinetic theory of gases suggests that the repulsion varies inversely as the molecular distance.

$$\left[p \propto \frac{1}{v}; \quad v \propto d^3 \therefore p \propto d^{-3}; \quad \frac{p}{n^2} = pd^2 \propto d^{-1} \right].$$

This, however, seems to disagree with the conclusion as to $\gamma_1 > \gamma_2$ if both are more than unity.

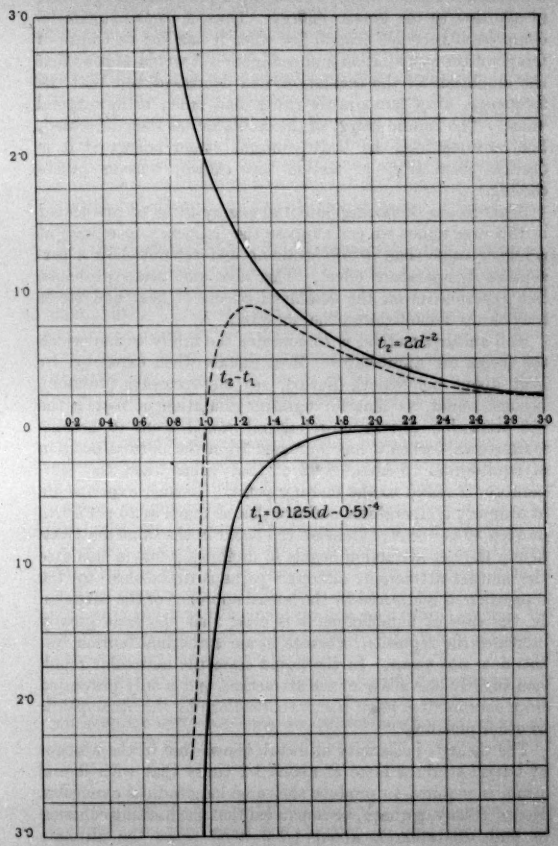


FIG. 3

The oscillation theory of repulsion is complicated by the fact that it will not simultaneously explain cohesion and temperature effects without including a potential energy store in addition to the kinetic energy. Thus, a piece of material compressed becomes heated, but when it has lost its excess of temperature by radiation and conduction it would appear that the amplitude of the oscillations had decreased, and that the frequency, after temporarily rising, had fallen to its original value. The kinetic energy of the oscillator has then decreased, but, nevertheless, the body contains strain energy if it is elastic. (Similarly, as Kelvin has shown, tension causes cooling.)

Some means of storing potential energy must be predicated in this case unless we can suppose the frequency to remain at a higher value than before straining and yet not show a perceptible temperature effect. This does not seem probable, but experiments on the combined effects of heat and stress may throw some light on the question.

Still another method is to consider the limits within which the forces are appreciable. Most metals when expanded by heat about one-fiftieth (linear), say one-sixteenth (volume), become liquid, so that for uniform dilatation of metals the net attraction (attraction-repulsion) would seem to become inappreciable when d has increased from the normal position of equilibrium to about 1·02 of that value when the temperature is raised to the melting point. Tension experiments at ordinary temperatures indicate that as much as 25 per cent., or even 50 per cent., extension can occur in the same materials before the net attraction begins to diminish; but in this case the mutual attraction, although perhaps diminished by the separation, is reinforced by the lateral approach of the particles. In the case of liquefaction it is clear that the heat greatly increases the repulsion, whereas in an isothermal tension test this does not occur. In the liquid state the molecular repulsion slightly prevails over the attraction, and is only prevented from causing dispersion (*i.e.*, vaporisation) by the atmospheric or other pressure.

The theory of elasticity in metals shows that in the absence of lateral strain a force of about 1·2 times that with lateral strain is required to produce the same longitudinal extension. Hence, it may, perhaps, be concluded that the limiting cohesion in pure dilatation is about $1/1{\cdot}2=0{\cdot}83$ times the ultimate stress in uni-directional tension.

When cooled to absolute zero from usual temperatures, steel has a contraction (linear) of about 0·003. It possesses a greater cohesion at very low temperatures, a fact which may be due to the reduction of the repulsion or to increase of the attraction. The former seems more probable. In any case, it indicates that repulsion is not wholly due to the kinetic energy of temperature oscillations, or that these are not zero at the absolute zero of temperature.

Apart from the suggestions of Boscovich, Kelvin and Tolver Preston, there do not seem to have been formulated any exact theories as to cohesion, but two recent theories of gravitation are of some interest in this connection—viz., Osborne Reynolds' "granular medium" and the electronic theory of Lorenz and Crehore.

In the former ("The Sub-Mechanics of the Universe," Cambridge University Press, 1903) Reynolds finds that cohesion and gravitation can be explained by the mutual attraction of "inequalities" (*i.e.*, deficiencies) in an universally diffused granular medium of "negative mass." He says :—

"Cohesion between the singular surfaces of negative inequalities results from the terms which were not taken into account in the first approximation which correspond to gravitation. These secondary terms involve the inverse distance to the sixth power, and therefore have a very short range, and so correspond to efforts of cohesion of the singular surfaces as well as surface tensions having no effect unless the singular surfaces, or molecules, are within a distance very small compared with the diameter of the singular surface." (I., §5.)

"In the analysis for the effort of attraction of negative inequalities the ratio of the volume of the grains absent divided by the volume enclosed by the singular surface has been neglected, and it is this simplification only which renders the law of attraction—as the inverse square—the law of attraction of the singular surface at a distance.

"But this in no way limits the variation of the stresses over those portions of the space in and between the parts of the two singular surfaces which are within indefinitely small distance of each other. Such limits can only be determined by taking into account the higher terms which have been neglected.

"This analysis I have not attempted. But it seems to me very important to notice this omission, as it appears that the attractions expressed by the higher powers of $1/r$, when the

surfaces are indefinitely near, must be of great intensity, so that, owing to sudden variations, the work done in separating the surfaces must be extremely small.

"These characteristics are those of cohesion and surface tension, and they promise to account by mechanical considerations for the hitherto obscure cohesion between the molecules as belonging to the attractions resulting from the finite diameter of the molecules divided by the curvature resulting from distortion, or we might say the complement of gravitation." (XIV., §227, slightly abridged.)

Reynolds' theory is open to objection on the ground of the apparent instability of the configurations which he deduces and seems to have been shelved, if not displaced, by the electrical theory of matter; but it is interesting to note his reference to the sixth power of the distance and the curvature effect.

Dr. Crehore has set forth an electrical theory of gravitation ("Electrical World," 1912; "Scientific American Supplement," No. 1,893, April 13, 1912, p. 238), in which, by assuming that the velocity of electrons in their orbits about the centres of atoms differs only slightly from the velocity of light,* he arrives at an expression for the mutual attraction of two atoms as follows:—

$$F=\frac{2q^2w}{K\pi v^2}\cdot\frac{e\,.\,e'}{R^2-a^2},$$

q=velocity of electron,

w=thickness of "magnetic sheet" due to the quasi-polarisation of each electron,

K=specific inductive capacity,

v=velocity of light,

R=distance from centre to centre of atoms,

a=radius of electronic orbit,

e and e' are the charges on the atoms.

By equating this to the gravitational formula, he finds that if $e=e'=4{\cdot}9\times10^{-10}$ electrostatic units and q almost exactly equals v, $w=1{\cdot}6\times10^{-36}$ cm., and that 99 per cent. of the induction (which causes the attraction by differential action

* This hypothesis does not receive confirmation from spectroscopic investigations.

at the two opposite points of the electronic orbits) is included within this thickness if $v-q=3{\cdot}9\times10^{-58}$ cm./sec.

He states that this equation is true for both great and small distances, but " It must be remembered that in deriving this formula the magnetic induction in the sheet accompanying the corpuscle was considered to be uniform throughout the volume of the positive sphere of the attracted atom. When the distance is very small the induction should be integrated throughout the volume, and the expression must not be used in case R is nearly equal to a."

If we assume that at absolute zero the molecular fields are almost in contact, the minimum value of R would seem to be $2a$ for the outside electrons. Remembering that the electrons are probably in several ring systems, $R=4a$ seems to be a fair estimate for the minimum. The expansion from absolute zero to ordinary temperatures being taken as 0·25 (an excessive value for many substances), a normal value, $R=5a$, may be appropriate for solids.

It is noteworthy that a curve of the form varying inversely as the sixth power of the distance is tangential to a curve of the form varying inversely as R^2-a^2 at a point $R=1{\cdot}225a$.

The differences between the curves are very small for a range of about $0{\cdot}05a$ above and below this value; but as the value $R=1{\cdot}225a$ would imply interpenetration of the molecular fields it would seem that either the sixth power rule does not apply or that the induction correction for small values of R makes an appreciable difference.

Crehore's formula has the great advantage that it enables the gravitational constant to be retained even at the smallest distances, the variation being simply due to the second term in the denominator or the expression

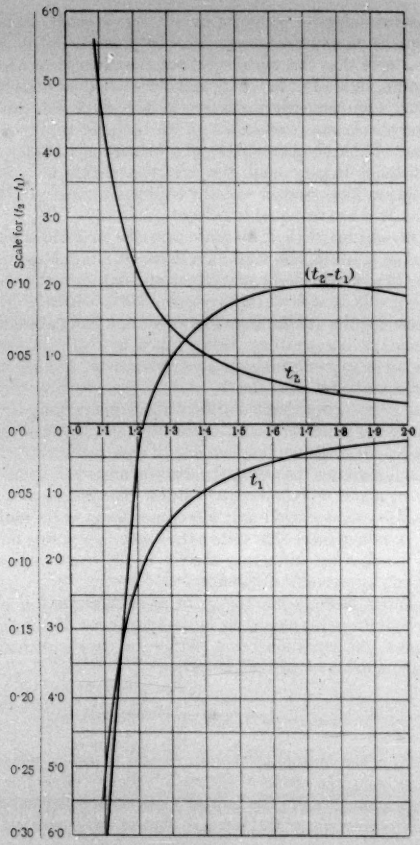
$$t_2=\frac{G\,.\,m^2}{(d^2-a^2)},$$

where t_2 is the mutual attraction, G and m are as before, and $d=R$, a being the radius of the mean effective electronic orbit.

The argument as to the indices γ previously mentioned still applies to functions of this form, so that we may, perhaps, write the repulsion

$$t_1=\frac{G_1m^2}{(d^{\gamma_1}-a^{\gamma_1})}$$

where γ_1 is greater than 2. A value of 3 for γ_1 gives a resultant



$$t_2=\frac{1}{d^2-1}$$

$$t_1=\frac{1\cdot676}{d^5-1}$$

$t_2=t_1$ when $d=1\cdot225$

FIG. 4.

curve, t_2—t_1, in which the net attraction persists at a high value through a range of several times a.

Larger values of γ_1 decrease the range of the net attraction, so that experimental information on that subject indirectly leads to γ_1 if Crehore's rule is true for the attraction.

Some information might, perhaps, be expected from astronomical research. Newcomb has discussed the possibility of the index 2 in the gravitational rule being slightly inaccurate as an explanation of certain irregularities in the perturbations of the inferior planets; but Brown's exhaustive investigations of lunar theory fail to confirm this supposition. Crehore's formula would lead to the conclusion that the force is very slightly in excess of the inverse square law (which would make the index less than 2 for any particular distance), but the difference becomes inappreciable when the ratio d/a exceeds 1,000.

The Effect of Atomic Weight on the Cohesion of Solid Elements.

A scrutiny of the tensile strengths compared with atomic weights for the elementary metals shows that there is no simple relation between the two quantities. There may possibly be a periodic one, in which case it should throw light on the occurrence of gases and liquids in the series. The very existence of these at normal temperatures and pressures indicates that the conditions of infra-atomic equilibrium vary with the number of electrons (*i.e.*, the atomic weight), and it is to be presumed that the cohesion is similarly variable among the different solids. Extremely high atomic weight does not give high cohesion, nor, on the other hand, does it appear in the elements very low in the scale. There would seem to be no doubt that the high atomic weights imply a large proportion of irreducible volume or a smaller number of atoms per unit mass, and this will doubtless affect both the attraction and repulsion; but, as the effective cohesion is the difference between these, no conclusion can be drawn.

The Effect of Molecular Combination on Cohesion.

It is in this direction that the most surprising results occur. Minute variations of constitution (as in the case of the iron and copper compounds) cause great differences, and, again, the combination of two gaseous elements may produce a solid, or two solids a gas. The heat produced by or absorbed in

combination doubtless suggests an explanation, but, anyway, it is clear that the mutual chemical attraction of the atoms can immensely exceed cohesion.

The Effect of Density.

Small increases of density unaccompanied by chemical change or perceptible structural change can produce great changes in cohesion. This may partly be accounted for by considerations of invisible crystalline structure, the surfaces of the crystals being brought into more intimate contact or gaining in mutual tenacity by dimensional relations; but it would appear to be true of matter with no super-molecular structure, although the question then arises whether, apart from crystalline structure, it is possible for matter to have two densities under the same conditions of pressure, temperature and chemical composition.

The specific gravity of metals of constant chemical composition varies considerably, but it seems probable that this variation is accompanied in all cases by crystalline change. Sub-crystalline structure, although oriented, is probably of almost maximum density. Super-molecular structure can only be of maximum density if the crystals completely fit one another or are pressed together so that all parts of the surfaces of each crystal are in contact with others except on the free faces of the solid. Pressure such as this must destroy the geometrical form of the solid, and possibly these damaged angles give rise to points of relative weakness.

XXXIV. *Some Novel Laboratory Experiments.* *By* F. W. Jordan, *A.R.C.S., B.Sc.*

Received April 25, 1915.

A Vacuum Flask Condensation Calorimeter.

It occurred to me that the errors in the condensation methods of measuring the latent heat of a vapour could be almost eliminated by adopting the following method: A steady stream of the vapour is passed for a certain time through a vacuum vessel containing an empty suspended bulb, and the mass of the condensed vapour is determined. In the second experiment the same steady stream of vapour is passed for the same time through the vessel, and condensed on the same bulb filled with water. If the experiments be performed under identical conditions, the difference between the masses of condensed vapour will be that required to raise the temperature of the water to that of the vapour. Errors arising from losses of heat by convection and radiation, condensation in the tubes and wetness of the vapour can in this way be almost eliminated. The capacity of the bulb can be made large, and the masses of condensed vapour can be determined with an ordinary balance which will carry a load of 500 gm., and is sensitive to 0·01 gm. As a test of the accuracy of the method it was decided to measure the latent heat of steam.

A sectional view of the steam trap and calorimeter as designed for this purpose is shown in Fig. 1. The steam from the boiler was delivered by an upward sloping wide tube to the steam trap, where the separation of the suspended water globules from the steam was effected by centrifugal force brought into play by the rapid motion of the steam through the spiral tube A. The steam was delivered from the boiler at the rate of 14 gm. per minute, and the calculated centrifugal force of several hundred times its own weight on a water globule was considered to be sufficient to drive it to the side of the spiral tube before the escape of the steam. The steam was finally delivered to the vacuum vessel by the tube B, which was surrounded by the coils of the spiral tube. The condensed water was drained off periodically by way of the tube C.

The mouth of the thermos flask was closed by a slightly conical shallow brass cup, fitted with an outer ring of india-rubber to render it steam-tight. The steam was sprayed evenly throughout the interior of the flask by the perforated brass tube D. The exit tube E was of sufficient width to allow

of the insertion of a thermometer, and to prevent spurting of the condensed vapour from the lip of the tube. The loss of heat from the stopper was minimised by a filling of cotton-wool and a covering disc of cork. The latter prevented absorption of the steam by the cotton-wool, and was removed before weighing.

The brass bulb H was suspended by a thread looped over the

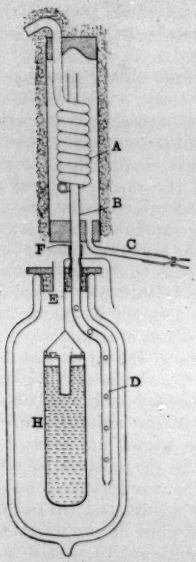


Fig 1.

inlet tube. This bulb could be filled by way of a small opening in the lid and closed by a small screw plug. The initial temperature of the bulb was obtained by lowering a thermometer through the exit tube into the re-entrant tube of the bulb.

In conducting an experiment the empty bulb was suspended inside the flask, and possible inequalities of temperature were removed by drawing a rapid current of air from the room

through the flask with a Lennox blower. The flask, together with the stoppers for the tubes, was then weighed. The thermometer was inserted into the bulb, and the initial temperature recorded after an interval of several minutes. The thermometer was then removed, and at a given instant the short indiarubber tube F was connected with the steam delivery tube. At the end of 10 minutes the supply of steam was stopped momentarily by pinching an indiarubber connecting tube; the vacuum flask removed, quickly stoppered and then weighed. It was found that this method of closing the mouth of the flask was quite satisfactory, and the weight remained perfectly steady to 0·01 gm. for many minutes. The flask was then emptied, dried and cooled by a rapid current of air to the temperature of the room. The bulb was almost filled with distilled water at room temperature, and the experiment repeated under almost identical conditions.

A preliminary experiment showed that the temperature of the full bulb attained an almost stationary value after the fifth minute, so that 10 minutes was an ample time to allow the bulb and the condensed steam to acquire the temperature of the steam. It was found as a result of a number of experiments under identical conditions that, in spite of the passage of a large amount of steam through the flask, the weight of condensed steam could be relied upon to about 0·01 gm. or 0·02 gm. The supply of steam could have been reduced after the second minute, but it was considered that the test would be more rigorous by maintaining a rapid supply of steam throughout the experiment. A reduction in the rate of supply of the vapour can be resorted to when waste of the vapour has to be avoided. If the quantity of condensed vapour is at all large it is better to catch the condensed vapour from the bulb in a shallow cup just beneath the bulb before allowing it to drip on to the bottom of the flask, where it can only be heated slowly by conduction from the upper surface. The time of passage of the vapour can then be reduced to six or seven minutes.

The thermal capacity of the flask and empty bulb may be determined to the nearest unit by measuring the steam condensed in the first minute in the calorimeter. In calculating the thermal capacity, loss of heat may be neglected, and an approximate value of the latent heat will be quite sufficient. The error arising from an omission of the correction for the thermal capacity of the vessel need not exceed 1 part in 300

in a suitably designed calorimeter, if the initial temperatures of this in the two experiments do not differ by more than a degree centigrade.

The preliminary experiments showed that as the dimensions of the apparatus were increased the final results became more concordant, and there is hardly any doubt that the latent heat of a vapour or the specific heat of a liquid, and in many cases that of a solid, could be measured to 1 part in 500.

The following dimensions of the vacuum vessel may serve as a guide to any alterations or improvements. The thermos flask was of the ordinary pint pattern; the diameter of the mouth was 2·7 cm., and the largest bulb that could be introduced had a capacity of 60 cubic cm. A thermos flask with a wider mouth and same internal volume would have given better results. The thermal capacity of the flask and empty bulb was about 23 calories. The weight of condensed steam on which the final result depended was about 9 gm. The steam was supplied at the rate of about 14 gm. per minute in most of the experiments; but this would have to be increased during the initial heating for a bulb of larger capacity. The initial temperature was read correctly to about 0·1°C. on an ordinary thermometer. The balance could carry a load of 500 gm., and the weight was measured to 0·01 gm. It may be remarked that an error of a few milligrammes in the set of weights, and this is a common fault, matters but little in this experiment.

The results obtained in six successive experiments for the latent heat of steam at 100°C. were 539·9, 539·3, 538·4, 540·1, 540 and 538·5, giving a mean value of 539·8, a result which agrees as well as can be expected with the accepted value of 540 at 100°C. in terms of the caloric at 20°C.

I have since shown, as the result of experiments with another steam calorimeter, to be described later, that the wetness of steam generated at atmospheric pressure is very small, and a separator of the ordinary pattern is quite sufficient to prevent condensed water in the delivery tubes from entering the calorimeter.

A Steam Calorimeter.

The loss of heat from the outer surface of the calorimeter can be prevented almost entirely by first raising rapidly the temperature of this surface almost to that of the steam by a sudden immersion in a current of steam, and then admitting this to the interior of the calorimeter. In order to test the accuracy of this method it was again decided to measure the latent heat of

steam and the apparatus as designed for this purpose is shown in Fig. 2.

The calorimeter A was made of hard thin sheet brass, and fitted with a screw-down lid, B. The latter was made steam-tight by a washer of moderately hard rubber. The steam when required was admitted to the interior of the calorimeter through the short inlet tube C, by removing the stopper in the exit tube D. The brass bulb E was constructed in the same way as for use with the vacuum vessel, and was suspended by a loop of thread from two small hooks on the lid of the calori-

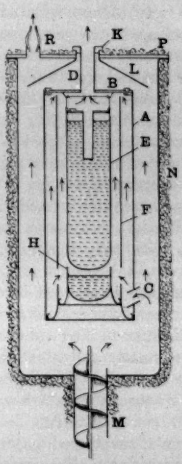


FIG. 2.—A STEAM CALORIMETER.

meter. The bulb E was shielded from any exchange of heat with the inner surface of the calorimeter by the radiation screen F. This consisted of a thin sheet copper vessel, perforated at the sides for the admission of steam and supported by three spring legs of thin wire on the base of the calorimeter. The upper edge of the screen was cut away to form three small projecting spikes and an alternative path for the steam. The pressure of the lid of the calorimeter against these spikes sufficed to hold the screen in position. The gain of heat by

the bulb from the radiation screen during its heating by the steam will involve a condensation of the steam on the screen, and this is included in the final weight. Thus, the introduction of this screen makes it possible to use a bulb of large thermal capacity. In the vacuum vessel the gain of heat by the bulb from the inner surface of the vessel is corrected by a condensation of the steam on this surface, and consequently this difficulty does not appear. The comparatively cool water condensed on the bulb was collected in a shallow copper cup, H, supported by three thin wire legs inside the radiation screen. The bulb was in this way almost isolated from exchange of heat with the inner surface of the calorimeter.

The calorimeter was suspended from the lid of the steam-jacket N by a circular nut of hard wood, K, screwed on to the exit tube D. A light umbrella of thin sheet copper prevented any cool-condensed water from falling on to the lid of the calorimeter. The steam before admission to the jacket was freed from suspended water by a rotary motion in the spirally divided inlet tube M. This spiral motion of the upward current of steam also contributed to a more rapid and uniform heating of the outer surface of the calorimeter and more consistent results were obtained with this device than in its absence. The additional mass of condensed steam, before the introduction of this device, did not amount to more than 0·1 gm., even when 200 gm. of steam was circulated through the calorimeter. Hence, the quantity of suspended water in the steam must have been very small, and this conclusion was still further verified by the fact that the progressive condensation in the calorimeter only amounted to a few centigrams in five minutes. Many indifferent results can be accounted for not so much by the presence of suspended water globules as by the transfer of condensed water along the surfaces of the tubes to or from the calorimeter. As an instance of this it may be noted that, during the course of an experiment on the latent heat of steam by a modified Berthelot's apparatus, it was found that 0·5 gm. of steam was condensed in the short inlet tube, whilst 24·5 gm. was condensed in the calorimeter in 14 minutes. The correction for the condensation in the inlet tube was, therefore, far from small, and under these conditions exceeded the correction for loss of heat from the calorimeter itself. In Berthelot's original experiment the effect of this condensation in the inlet tube was minimised by a rapid flow of the vapour and absorption of heat from the ring-burner.

In conducting an experiment the empty bulb E was suspended from the cover, the calorimeter was fitted together as shown in the figure, and weighed together with the stoppers for the inlet and exit tubes. The calorimeter was then attached to the lid P of the vapour jacket and a rapid stream of air was drawn through the calorimeter by a Lennox blower to equalise the temperatures of the various parts. The temperature just before immersion in the vapour jacket was taken with a thermometer inserted into the bulb E. During these operations steam was got up in the boiler and passed through the jacket for two or three minutes. The exit tube D of the calorimeter was stoppered and the lid of the vapour jacket, supported by a cord handle, was held in readiness. The temporary lid of the jacket was removed at a given instant, and the calorimeter was suddenly immersed in the steam. At the end of 20 seconds the jet of steam from the exit tube R in the lid was well developed, and this was considered to be the best instant for diverting the greater part of the steam from the outside to the inside of the calorimeter. This was effected by removing the stopper in the exit tube D of the calorimeter, and reducing the escape of steam from the exit tube R in the lid of the vapour jacket by inserting a small nozzle. The steam in every case appeared at the exit tube D of the calorimeter a few seconds after this adjustment. At the end of two minutes the supply of steam was reduced to about a third by lowering the bunsen flame. This adjustment can be conveniently and repeatedly accomplished by fastening a long arm to the handle of the gas tap, and swinging this from one stop to another. At the end of six minutes the exit tube D was stoppered; the calorimeter was quickly withdrawn from the jacket, and its inlet tube stoppered immediately. The calorimeter was allowed to cool and then weighed. The existence of any leaky joint was rendered evident by an increase of weight, and every care was taken to avoid this defect. The rubber stoppers and washer answered perfectly well in this respect, and there was no trouble from leakage. Air was finally admitted and the calorimeter weighed.

The choice of these intervals of time was the result of a number of trial experiments. The first interval of 20 seconds was selected for the reason that an error of a second or two in its estimation affected the weight of condensed steam only to a slight extent. The next interval up to the end of the second minute was the result of an observation on the rise of temperature of the bulb when filled with about 100 cubic cm.

of water. The steam was supplied initially at the rate of about 20 gm. per minute, and the temperature of the bulb rose to about 40°C. at the end of the first minute, and 90°C. at the end of the second. The lag of the thermometer made it impossible to be precise in stating the first of these temperatures. The temperature was practically steady at the end of the fourth minute, and hence the selection of 6 minutes was considered ample to allow the bulb to acquire the temperature of the steam.

In the second experiment the bulb was nearly filled with water, and the experiment repeated under almost the same conditions. The additional amount of condensed steam was considered to be that required to raise the temperature of the water to that of the steam.

The correction to be applied for the thermal capacity of the empty bulb and radiation screen amounted only to 1 part in 1,000, when the initial temperatures of the calorimeter in the two experiments differed by 1°C. The corrections for air displacement applied to the weights affected the weights of the condensed steam and water in the bulb equally, and can, therefore, be omitted.

The following data in connection with this apparatus may serve as a guide to any improvements and alterations. The height of the brass calorimeter was 18 cm., its diameter 6·6 cm. and its weight 250 gm. The brass bulb had a capacity of about 105 cubic cm. of water, and condensed under ordinary conditions about 15 gm. of steam. The weight of the interior parts was about 100 gm., and the combined thermal capacity about 9 calories. The weight of steam condensed in the calorimeter with the empty bulb was about 1·7 gm. in six minutes, and the weights obtained in successive experiments under the same conditions agreed to about 0·02 gm.

The results obtained in seven successive experiments for the latent heat of steam at 100°C. in terms of the calories at 20°C. were 539, 539·9, 540·1, 538·8, 539, 539·6 and 539, giving a mean value of 539·3, a result which differs but little from the accepted value of 540 at 100°C. These results justify the use of this apparatus as a steam calorimeter when an accuracy of not more than 1 part in 400 is aimed at.

A Differential Telephone Receiver.

The use of this instrument for electrical measurement is not new, but it may be of some interest to know that the ordinary

receiver can be used in conjunction with a simple differentially-wound transformer to serve the same purpose.

A core, composed of a bundle of soft iron wires or a soft iron rod, is overwound differentially with two primary coils P, P, Fig. 3, and the secondary coil S, which is connected to the telephone receiver T, is arranged so that it can slide over the primary coils P, P. The advantage of this construction is that it is possible, by moving the secondary S to a particular position, to obtain almost complete silence in the receiver T, when the primary coils P are traversed oppositely by equal alternate currents. It would be difficult to wind the core of an ordinary receiver with two coils so that they were exactly differential.

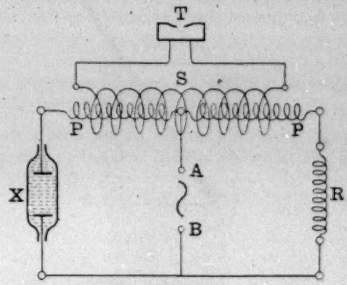


FIG. 3.—A DIFFERENTIAL TELEPHONE RECEIVER.

The resistance of an eletrolyte can be measured with the aid of this instrument by the differential arrangement shown in Fig. 3. The terminals of the secondary coil of a small induction coil are connected to A B, and the current is divided between one of the primary coils P in series with the electrolyte X and the other coil P in series with a dial box of resistance coils R. The telephone receiver will be silent when R=X, and this method is quite a sensitive one for this measurement.

The Thermal Conductivity of a Narrow Bar by a Modified Gray's Apparatus.

In Gray's apparatus the narrow bar is soldered at one end into the side of a copper boiler and at the other into a copper sphere. The heat conducted along such a narrow bar, when one end is heated by boiling water and the other cooled by contact

with the copper sphere, is so small that the temperature of the copper sphere has to be observed in some cases for 40 minutes. In order to increase the rate of transmission of heat four parallel metal bars, each about 2·5 mm. diameter and 8 cm. long, were soldered at their lower ends into a copper steam pipe A and at their upper ends into a thick copper plate B, 4 mm. thick, forming the base of the water calorimeter. The latter was chosen for two reasons, firstly, because it is more convenient and in accordance with practice, and, secondly, because an exact knowledge of the specific heat of the copper of the calori-

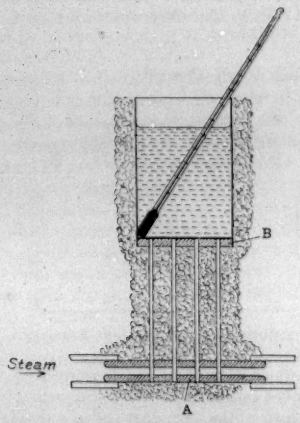


FIG. 4.—THERMAL CONDUCTIVITY OF A NARROW BAR BY A MODIFIED GRAY'S APPARATUS.

meter is not required. The use of four narrow bars, and not a single one of the same total cross-section, is necessary in order to ensure a large area of transmission of heat to the water, and thus minimise the difference of temperature between the latter and the upper ends of the bars. A measurement with thermo-junctions attached to the bars showed that the error, committed in assuming that the terminal temperatures of the bars were identical with the steam and water, if this be kept stirred, is of the order of 4 per cent. The error in some experiments on thermal conductivity, arising from the use of several thermo-

meters, for the measurement of temperature gradient and conducted heat, may amount to 2 or 3 per cent. if the thermometers have not been previously compared and corrected for stem exposure. Hence, in an elementary experiment it is justifiable to make an assumption like the foregoing and reduce the measurements as much as possible. The steam pipe, bars and calorimeter were lagged with cotton wool, and exchanges of heat between the calorimeter and its surroundings can be compensated by Rumford's method.

The final expression for the thermal conductivity approximates to the ideal one that holds for steady flow conditions, and the experiment can yield almost as good a result as other experiments in the absence of refinements and troublesome corrections.

Method of Measuring Poisson's Ratio for a Rectangular Beam or Lath.

The beam is bent into an arc of uniform curvature by equal couples applied to the two ends as shown in the figure. Let

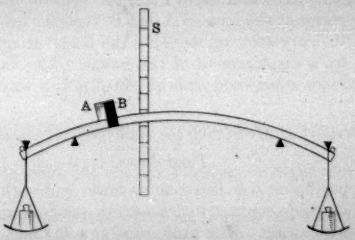


FIG. 5.—METHOD OF MEASURING POISSON'S RATIO FOR A RECTANGULAR BEAM.

R and r be the elastic and anti-elastic radii of curvature respectively. Then Poisson's ratio

$$\sigma = \frac{R}{r}.$$

The elastic radius of curvature is easily calculated from the elevation of the centre of the beam, and the distance between the knife-edge supports. To determine the anti-elastic radius of curvature I attached, at a little distance from one of the knife edges, two strips of plane mirror glass, A B, to the ver-

tical sides of the bar, so that viewed from the front of the figure their surfaces overlapped. A vertical illuminated scale was set up to one side of the common normal to the two mirrors at a distance of about a metre from the beam. A telescope was directed towards the mirror A, so that an image of the scale S formed by two reflections at the mirrors could be seen in the field of view. The anti-clastic curvature was rendered in this way quite evident and easily measurable. The mirrors were fixed near to the knife-edge support to minimise their elevations on bending the beam.

Let s be the change of scale reading on bending the beam, d the distance of mirror B from the scale, and $2b$ be the distance between the mirrors. Then the change of angle between the mirrors is $\frac{1}{2}\frac{s}{d+b}$, and the anti-clastic radius of curvature

$$=1\frac{(d+b)b}{s}.$$

If the ends of the beam be gripped in the two hands and an image of the horizontal bar of a distant window be viewed after two reflections at the mirrors A and B, then the existence of the anti-clastic curvature on bending the beam at once becomes evident by a displacement of the image. The value of Poisson's ratio for a boxwood scale obtained in this way was 0·29.

A Method of Measuring the Coincidence Period of Two Pendulums.

Each pendulum was fitted at its lower end with a platinum wire which made contact with a narrow pool of mercury at the centre of its swing. The knife-edge support and pool of one pendulum were connected electrically in series with the pool and knife-edge support of the other and with a cell and a telephone receiver. Thus, a click in the telephone was heard whenever the pendulums made contact simultaneously with the pools of mercury. The clicking in the telephone will persist over a short interval of time, which will depend on the widths of the pools of mercury and the amplitudes of the pendulums. This interval in one case was 10 seconds when the coincidence period was about 7 minutes. The coincidence period can in this way be measured correctly to one or two seconds, and the change of period of a Kater pendulum with diminishing amplitude can be readily detected. This method

appeals to students on account of its simplicity, and the only unfortunate feature of the device is that very little is left to test the skill of the student.

Achromatic Interference Fringes.

The number of visible fringes in white light produced with the aid of a Lloyd's mirror can be considerably increased by interposing a simple grating, G, of about 20 or 30 lines to the millimetre between the slit S and the Lloyd's mirror M (Fig. 6).

The mirror M consisted of a strip of good plate-glass mirror 30 cm. long and 5 cm. wide, with the silvered surface exposed to the light from the slit. The varnish at the back of this mirror was removed with turpentine and the film of yellow-scaly oxide underneath this by caustic potash. The silver surface was then washed, dried and polished with a pad of chamois leather. The vertical edges of the mirror were

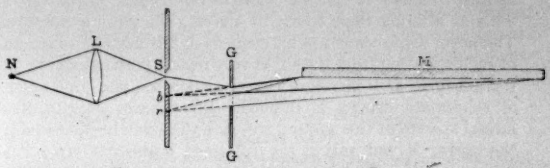


FIG. 6.—ACHROMATIC INTERFERENCE FRINGES.

blackened to obstruct the transmission of light through the glass.

The grating was made by ruling the lines on a smoked-glass plate with a simple dividing engine in the following way: The anvil of a micrometer screw gauge was removed and the handle was firmly clamped to the edge of a table. The rounded end of a rod about 4 ft. long was inserted in the place of the anvil, and pressed against the screw by an elastic band. A specially pointed needle was fixed to the other end of the rod, and this was guided with a cork handle during the ruling. The greater part of the weight of the rod was supported by a counterpoise to reduce the pressure on the needle point and minimise the bending of the rod. The smoke on the glass plate was made more than usually adherent by flowing over it a weak solution of shellac in alcohol. To obtain the best results the first order spectra should be of predominant intensity and this was

effected, in accordance with theory, by making the opaque and transparent elements of the grating of nearly the same width.

The slit S was adjusted to a width of about 0·05 mm., and was illuminated by the light from a Nernst filament N. The mirror M was adjusted so that its surface was parallel to and in a line with the slit, in which position only the diffraction fringes were observable with an eye-piece at a distance of about 50 cm. from the slit. The grating G had about 30 lines to the millimetre, and all but a width of a millimetre was screened from the light from the slit. The grating was arranged at a distance of about 5 cm. from the slit, with the rulings nearly parallel to the length of the slit, and just in front of the mirror M. The best width and position of the narrow grating is that in which it will just admit the first order spectrum to the mirror M. The direct white light from the grating will then fall to one side of of the mirror and interfere but little with the distinctness of the fringes. If the distance of the grating from the slit be increased to about 10 cm. the colour of the fringes can be varied without altering their width by moving the grating laterally. The lineal displacement of an element of the vertical spectrum of such a grating from the slit is very nearly proportional to its wave-length, and therefore the condition for the production of achromatic fringes is fulfilled. Interference between the lateral spectra of this grating will give the same result without the mirror M, but only if the diffracted beams overlap. For instance, in the above experiment, if the mirror M is removed, the achromatic fringes will not become visible until the width of the grating is such that, in the neighbourhood of the eye-piece, the diffracted beams overlap. Moreover, it is necessary that the lines of the grating should be more strictly parallel to the slit, whereas with Lloyd's mirror this adjustment need not be so exact.

Interference between spectra of higher order than the first will occur as the width of the grating is increased, but since these spectra are comparatively faint, the interference effects are scarcely observable, except with monochromatic light. The direct light from the grating interferes somewhat with the distinctness of the fringes, but not to the extent that one would at first imagine. The whiteness of the fringes is also impaired by an admixture of diffracted coloured pencils of light, but this adds to the colour effect and appearance of the fringes.

The direct light can be made very feeble by using a transmission grating in which the elements are of equal width, and

alternate elements are made of a thin film of gelatine of a thickness sufficient to retard the light by half the average wavelength of the most intense portion of the spectrum. I have not been able to procure such a grating, but its effect can be seen by viewing a sodium flame on a batswing burner edgewise through a small portion of the edge of a phase-reversal zone plate. The direct image of a white source of light will be coloured according to the element of the spectrum that is suppressed by the phase-reversal. The direct image will be very faint and the first order spectrum will appear almost as bright as the direct image given by a smoked-glass grating. Such a grating for the particular purpose of this experiment ought to give better results than the grating on smoked glass.

It may also be remarked that the structure of a phase-reversal grating which is almost invisible in diffused light, becomes easily visible under the conditions of this experiment owing to the appearance of the fringes formed by interference between the lateral spectra. If the slit be widened these fringes would vanish, and the grating would present almost the same appearance as in diffused light. The same remark applies to the case when the grating is viewed through a microscope.

ABSTRACT.

Condensation Calorimeters for the Measurement of Latent and Specific Heats.

(*a*) It is shown how an ordinary vacuum jacket flask can be converted into a suitable condensation calorimeter. Errors arising from loss of heat and wetness of the vapour are almost eliminated by making two experiments.

(*b*) Another condensation calorimeter is constructed on a different plan with a view to the elimination of the same errors.

The Thermal Conductivity of a Narrow Metal Bar.

Gray's apparatus for the measurement of the conductivity of a narrow bar has been modified for the purpose of rendering the loss of heat relatively small and reducing the time taken in the measurement, thus introducing the experiment to the elementary student.

The Measurement of Poisson's Ratio for a Rectangular Lath.

Two mirrors are attached to opposite sides of the bent lath for the measurement of the anti-clastic radius of curvature, and Poisson's ratio is then deduced from the observations in the usual manner.

A Method of Measuring the Coincidence Period of a Kater and a Clock Pendulum.

The Kater and the clock pendulums are electrically connected so that when at the centres of their swings a momentary current passes through a telephone receiver.

A Differential Telephone Receiver.

An ordinary receiver is connected to the secondary of a simple differentially wound transformer, and it is thus converted into a differential receiver for the purpose of electrical measurement.

Experiment on Interference Fringes.

The fringes produced by a Lloyd's mirror in white light are nearly achromatised by a simple grating on smoked glass. Other suggestive interference effects with the grating are also described.

DISCUSSION.

Mr. F. E. Smith asked why in the thermal conductivity experiment four rods were used instead of one of four times the cross-section ?

Mr. H. Moore admired the ingenuity of the pendulum experiment, but thought that to make an experiment too simple and free from corrections made it much less instructive to students. He thought the eye and ear method of observing coincidences was one in which students should be trained as much as possible.

Mr. Jordan, in reply to Mr. F. E. Smith, said that the use of four thin rods rather than one thick one was advisable, as there was a smaller temperature drop between the end of the rod and the calorimeter when the former was thin than when it was thick.

XXXV. *The Coefficient of Expansion of Sodium.* By Edgar A. Griffiths *and* Ezer Griffiths, *M.Sc.*

Received May 20, 1915.

The Coefficient of Expansion of Sodium.

Introduction.—The experiments recorded in this Paper were undertaken with the object of ascertaining the variation in the coefficient of expansion of sodium in the vicinity of its melting point.

The datum is of interest in its bearing on the modern theories of specific heat since the square of the coefficient of expansion is one of the factors which enter into the calculation of the difference between C_p, the atomic heat when the metal is free to expand and C_v, the atomic heat at constant volume.

Theoretical formulæ for the representation of C_v and its variation with temperature have been given by various writers, notably Einstein and Debye.

Einstein based his reasoning on the quantum-theory of energy and the assumption that the atoms in a solid oscillate about fixed positions, the atomic system being equivalent to a number of Planck's resonators, while Debye proceeds on the supposition that the heat vibrations depend on the elastic forces in such a way that the longest heat waves are identical with elastic vibrations.

On account of the experimental difficulties of the problem, direct determinations of C_v have not been attempted in the case of solids, and consequently to test the theoretical formulæ it is necessary to deduce C_v from a knowledge of C_p and the thermodynamic relationship * between C_p and C_v.

In the case of sodium it has been shown that C_p increases very rapidly with the temperature as the melting point is approached, rising from 6·5 at 0°C. to 7·5 at 96°C.†

Since the limiting value of C_v for an element at high temperatures is approximately 6, it seemed of interest to test the

* $C_p - C_v = a^2 e v A t$.

Where a is the coefficient of cubical expansion.
e ,, ,, volumetric elasticity.
v ,, reciprocal of the density.
A ,, atomic weight.
t ,, absolute temperature.

Note.—In Phil. Trans. Roy. Soc. A. 518, p. 339, Vol. 214 (1914), the atomic weight coefficient (A) was inadvertently omitted in writing down this relationship between C_p and C_v.

† "Proc." Roy. Soc., p. 561, Vol. LXXXIX. (1914).

validity of the theoretical formulæ for a metal near its melting point by determining the data necessary for the calculation of $C_p - C_v$.

Theory of Method.

On account of its plastic nature and the tendency to oxidise, the ordinary methods for the determination of the coefficient of expansion cannot be applied to sodium.

The method described in this Paper was devised to overcome these difficulties, and the principle upon which it is based is to measure the difference in expansion of a volume of sodium and an equal volume of glass or of quartz by differential weighing under oil at various temperatures.

The essential features will be seen from an examination of Fig. 1.

Let W be the true mass of the sodium.

w „ „ true mass of the containing envelope.

W_1 „ „ apparent mass of the sodium in oil at temperature t_1.

w_1 „ „ apparent mass of the envelope in oil at temperature t_1.

X „ „ true mass of the compensating globe.

X_1 „ „ apparent mass in oil at temperature t_1.

Volume of sodium at temperature $t_1 = \frac{W - W_1}{d_1}$, where d_1 is the density of the oil at t_1.

If K is the coefficient of cubical expansion of sodium, volume at temperature t_2,

$$= \frac{W - W_1}{d_1}[1 + K(t_2 - t_1)].$$

If g is the coefficient of cubical expansion of the glass, volume of glass envelope at temperature t_2,

$$= \frac{w - w_1}{d_1}[1 + g(t_2 - t_1)].$$

Volume of compensating globe at temperature t_2

$$= \frac{X - X_1}{d_1}[1 + g(t_2 - t_1)].$$

If Z is the additional mass required to produce equilibrium when the temperature is changed from t_1 to t_2, then Z/d_2 is the volume of oil displaced by the expansion of the sodium and its

envelope relative to the compensator, where d_2 is the density of the oil at t_2.

Hence,

$$\frac{Z}{d_2}=\frac{W-W_1}{d_1}[1+K(t_2-t_1)]+\frac{w-w_1}{d_1}[1+g(t_2-t_1)]$$
$$-\frac{X-X_1}{d_1}[1+g(t_2-t_1)].$$

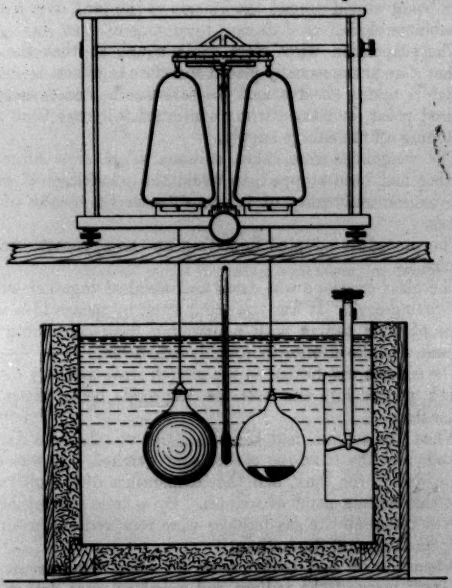


FIG. 1.

Now, by the preliminary adjustment,

$$W-W_1+w-w_1=X-X_1.$$

$$\therefore \frac{Z}{d_2}=\frac{(t_2-t_1)(W-W_1)(K-g)}{d_1}.$$

Hence,

$$K-g=\frac{Zd_1}{(t_2-t_1)(W-W_1)d_2}$$

Description of Apparatus.

In the actual observations the glass envelope containing the sodium was suspended from the left-hand pan of a 10 cm. beam balance while the sealed glass bulb was suspended from the right.

The tank contained transformer oil which could be heated to any desired temperature by means of a resistance coil, the wire being wound around the outside of the tank over a sheet of asbestos cloth.

The suspension wires were made equal, so that the two globes were symmetrically placed and close together.

Before taking observations the bath was heated to near the desired point, and the stirring continued for some time after switching off the energy supply.

The weighings were taken as soon as possible after the stirring had been stopped, to avoid the possibility of errors due to non-uniformity of temperature in the region of the globes.

The procedure adopted in setting up the apparatus was as follows :—

The glass envelope was dried and weighed together with a well-fitting cork. It was then connected by means of a short piece of wide tubing with a distilling flask containing dry sodium, and the interior thoroughly exhausted.

The sodium was heated to the molten state and, by careful tilting, poured into the envelope, the oxide being left in the larger flask.

When the sodium had solidified, the vessels were disconnected, and the envelope corked and weighed. It was then suspended in the tank, and the temperature of the oil raised past the melting point of sodium. By a little stirring of the molten metal all the gas bubbles were removed and a film of oil formed around the sodium.

When the oil had cooled down to room temperature the apparent mass of the sodium and envelope were determined when immersed in the oil and hence the volume displaced.

The volume of the compensating bulb was chosen to be equal to that of the sodium and envelope within a few cubic centimetres and then an additional small bulb was blown to bring the adjustment within a fraction of a cubic centimetre.

The final adjustment was made by fusing on minute bits of glass to the long sealing-off stem.

By a few trials it was possible to make the loss in weight in

oil of the compensator the same as that of the sodium and envelope to within very narrow limits. In the present case the loss in weight on immersion was adjusted to a residual inequality of 0·02 gm.—*i.e.*, 0·023 cubic cm., and since the displacement of the sodium was 250 cubic cm. the compensation was correct to better than one part in 10,000.

Results.

As the experiments were regarded as being merely of a preliminary nature, it is unnecessary to record all the data obtained, and the following observations are selected at random from the records :—

Weight of sodium in air	=242·868 gm.
,, ,, oil	= 29·342 ,,
,, of oil displaced	=213·526 ,,
Density of the oil at 17 deg.	= 0·854 ,,
Hence volume of the sodium	=250·03 cubic cm.

TABLE I.

Column I. Temperature of the bath.
II. Additional weight to produce equilibrium, the balance being counterpoised at 17 deg.
III. Relative increase in volume of the sodium.
IV. Increment per cubic centimetre.
V. Absolute increase of unit volume at 17 deg. after correction for the glass.

I. Temperature.	II. Equilibrium weight.	III. Relative volume increase.	IV. Increment per cubic cm.	V. Absolute increase.
25·34	0·295	0·348	0·00139	0·00161
43·11	1·057	1·261	0·00504	0·00572
61·13	1·790	2·162	0·00865	0·00979
70·90	2·158	2·625	0·01050	0·01190
91·34	3·025	3·737	0·01495	0·01688
100·21	8·625	10·728	0·04291	0·04507
76·90	2·428	2·966	0·01186	0·01342
54·36	1·505	1·810	0·00724	0·00821
23·12	0·240	0·282	0·00113	0·00129
80·65	2·560	3·137	0·01255	0·01420
96·46	3·190	3·956	0·01582	0·01789
98·12	8·530	10·590	0·04235	0·04446

Some of the observations are shown graphically in Fig 2.

The value of the coefficient of cubical expansion deduced from these observations is 0·000226 per deg., and the change in volume on passing from the solid to the liquid state amounts to 2·57 per cent.

The experiments also give the value 0·971 for the density of sodium at 17 deg., and the value 0·930 for the molten metal at its freezing point.

Calculation of $C_p - C_v$.

Taking the value $6{\cdot}5 \times 10^{10}$ for the coefficient of volumetric elasticity as determined by Richards (Jour. Chem. Soc., 1911), and calculating the difference between C_p and C_v for the

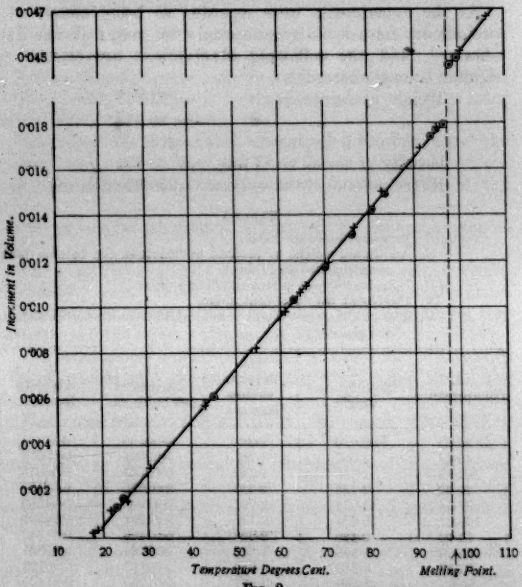


FIG. 2.

temperature of the melting point (370·6 deg. abs.), we have $C_p - C_v = 0{\cdot}70$.

This is the maximum value possible for the difference since the coefficient of volumetric elasticity probably decreases with the temperature and Richard's value was obtained at room temperature. Also the coefficient of expansion of all metals

hitherto investigated decreases very rapidly at low temperatures, and the above experiments have shown that in the vicinity of the melting point the coefficient of expansion of sodium is sensibly constant.

Hence, the value of C, deduced from experimental data exceeds the theoretical limiting value by at least 13 per cent.

The experiments above described were intended as a preliminary series to test the method and to ascertain whether the coefficient of expansion presented any anomalies near the melting point.

In utilising the method for further work it would be advisable to use fused quartz as material for the counterpoise, since its coefficient of expansion is small and sufficiently definite to render unnecessary an independent determination of its value for the compensator.

ABSTRACT.

The thermal expansion and increase in volume on liquefaction of sodium were determined by a method based on the following principle :—

The difference in expansion of a volume of sodium and an equal volume of glass (or quartz) was measured by differential weighing under oil at various temperatures.

A volume of about 250 c.c. of sodium was suspended from one arm of a short beam balance and a weighed glass bulb of equal displacement from the other arm.

The experiments show that sodium expands quite uniformly with the temperature up to its melting point.

The value 0·000226 was deduced for the coefficient of expansion. In changing from the solid to the liquid state, an increase of 2·57 per cent. occurs in the volume.

DISCUSSION.

Mr. F. E. Smith expressed his admiration for the method employed and for the order of accuracy with which the observations appeared to have been made. Would it have been possible to obtain sufficient accuracy with the sodium in an ordinary dilatometer, the remainder of which contained oil, as was often done in the case of other metals ?

Mr. A. Campbell asked if there had been no trouble from convection in the oil at the higher temperatures. Had the authors noticed any hysteresis effect in the expansion of the glass globe such as the behaviour of a thermometer bulb would lead one to expect ?

Dr. C. Chree stated that the two curves which Mr. Griffiths had drawn on the board representing the theoretical and experimental results were very similar in form, which might indicate that the theory had something in its favour. Had the coefficient of elasticity been determined for the actual specimen used, or were general results assumed ? He believed that the elasticity varied considerably in different specimens.

Mr. J. Guild asked what precautions had been found necessary to avoid difficulties due to the very uncertain capillary effects where the suspensions enter the oil. Was the compensating bulb evacuated ? If it contained air at atmospheric pressure when cold, the pressure would

increase by about a third of an atmosphere at 100°C., which would produce an appreciable alteration in the volume of the compensator. The neglect of a correction for this pressure effect would be to give less than the true value for the expansion of the sodium, and, therefore, of $C_p - C_v$. This might account for the discrepancy between the observed and theoretical values of C_v, and it seemed advisable in future work either to evacuate the compensator or to determine its total expansion from all causes by a blank experiment.

Mr. S. D. CHALMERS asked if a check was made to see if the volume of the bulb was the same before and after the experiments. Had the authors tried if Lindemann's formula would agree with their results?

Mr. GRIFFITHS, in reply, said that in the dilatometer method the difference in expansion of sodium and oil would be measured. The coefficient of expansion of oil is about 10 times that of a metal, and would have to be determined with considerable accuracy. The method described was intended to measure the expansion relative to that of fused quartz, of which the coefficient is about 1/200 that of a metal and is accurately known. The expansion of the oil only affects the ratio of the densities at the two temperatures. The effect of convection was greatly reduced by the symmetrical disposition and closeness of the bulbs. No secular changes were found, the results obtained before and after heating to the higher temperatures agreeing (*see* Table I. and Fig. 2). With quartz bulbs no secular effects would be possible. The formulæ of Einstein and Debye were able to represent the general form of the atomic heat-temperature curve, but systematic divergencies from the experimental results occur and the theories fail to account for the abnormal increase near the melting point. Nernst and Lindemann's formulæ gave a curve very similar to Debye's (*see* "Phil. Trans." R.S., A., Vol. 214, page 343, 1914). He had assumed Richards' value for the elasticity. The suspensions were of fine tinned steel wire. As the oil wetted the surface the capillary effect would probably be constant, and the same for both wires. The compensator was heated to about 200°C. when sealing off and, being stout walled, would be little affected by the changes of pressure of the residual air when heated. The expansion of the compensator was checked by weighing it alone in water at different temperatures.

XXXVI.—*Notes on the Calculation of "Thin" Objectives.* By T. SMITH, *B.A.* (*From the National Physical Laboratory.*)

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Synopsis.

FORMULÆ are given for the calculation of the curves of thin achromatic objectives, when two aberration conditions have to be satisfied. A two-lens objective satisfying the conditions as a rule cannot be cemented, because the curvatures of the inner surfaces are unequal. Formulæ are given for the calculation of a triple objective satisfying the conditions, but with only two glass-air surfaces. Numerical examples are worked out illustrating the application of the formulæ to the calculation of an astronomical object glass, and the kinds of glass are determined with which a cemented doublet satisfies the conditions for this case.

Aberration Theory of Thin Lens.

A lens system which approximates to the mathematically "thin" lens would merit considerable attention were there no uses to which it could be put but as the object glass of a telescope. Its usefulness is not, however, limited to such simple instruments, but very complex systems, such as periscopes for submarines, may be regarded as composed of a number of thin lenses. In the design of these more complicated instruments it is important to be able to arrive without excessive labour at a system which produces the desired optical effect, and these notes deal with one aspect of this problem. The positions occupied by the various lenses, their focal lengths and apertures are not here considered; they will usually be fixed by considerations of the field of view that is desired, the required magnifying power, and the mechanical limitations imposed by the dimensions of the body into which they are to be placed. Neither will the part to be taken by the several constituents in rendering possible the required perfection of the complex system be determined. It is assumed that both these problems have been solved, and that all that now remains to be done is to find how, with given kinds of glass, each subordinate system must be constructed to produce the effect that the previous investigations have shown to be necessary.

The first order aberrations of any lens system symmetrical

about an axis are determined by six quantities. In the notation used by the author these are expressed in the form of ratios, so that they are independent of the dimensions of the system. They are denoted by A, B, C, ϖ, B′, A′. When a beam of light from an infinitely distant object on the axis of the lens travelling in the positive direction is brought to the same focus by all zones of the lens, A is equal to zero. When this condition is satisfied by light from an infinitely distant object slightly off the axis of the lens, as well as from the point on the axis, B is equal to zero. $A'=0$ and $B'=0$ are the corresponding conditions when the parallel beams of light approach the lens from the negative side. The quantities C and ϖ have the same meaning for light travelling in the positive direction as for that travelling in the negative direction.

The general problem of designing a system to have arbitrarily assigned values for the six coefficients A, B, C, ϖ, B′, A′ is probably insoluble algebraically. Were such a solution obtainable it would almost certainly be too cumbersome for numerical use. When, however, the system may be treated as thin, that is to say when the thicknesses and separations of all the component lenses are so small that they may for a first approximation be neglected, the formulæ for these coefficients take simple forms, and many problems connected with thin lenses may be reduced to the solution of quadratic equations. The values for the curvatures of the lens surfaces obtained in this way are often—as in telescope objectives—in agreement to a high order of accuracy with the values obtained, when the thicknesses of the lenses are taken into account. In other cases appreciable modifications may be necessary, but much time can be saved by finding the general character of each lens before trigonometrical or other exact methods of computation are employed.

In all thin systems in which the first and last media through which the light passes are the same—for instance any lens in air—the six aberration coefficients are connected by three identical equations, viz.:

$$A+C=2B+1,\ B+B'=2C+\varpi,\ C+A'=2B'+1. \quad (1)$$

There are thus in any thin system only three degrees of freedom instead of six, as in the general case. Moreover, in practice, ϖ lies between very narrow limits, and in the majority of thin lenses it is definitely fixed when the kinds of glass to be employed are known. There are therefore only two conditions

which can in general be satisfied by a thin lens. In the numerical illustrations which follow, the conditions selected are those required in an astronomical objective, viz., that the outer zones of the lens shall have the same position for the principal focus, and have the same focal length as the central zone, or stated in technical terms, the system is to be free from central spherical aberration, and from coma for an infinitely distant object. The algebraic formulæ given below can, of course, be applied to find a system having any desired values of the aberration coefficients consistent with equation (1). The problem already mentioned of finding the properties each thin system is to have is supposed to have been solved in the form of assigning to these coefficients values consistent with these three conditions.

The coefficients by which the aberrations are measured can easily be calculated for a thin system having any number of spherical refracting surfaces. Suppose the number of these is n; let them be identified by the numerals 1, 2, 3, . . . , λ, . . . n where the numbers indicate the order in which the surfaces are met by a beam of light traversing the system in the positive direction. Let their curvatures be denoted by $R_1, R_2, R_3 \ldots, R_\lambda, \ldots R_n$, the curvature of any surface being positive if it is convex to incident light travelling in the positive direction. The powers of the surfaces will be denoted by $K_1, K_2, K_3, \ldots, K_\lambda \ldots K_n$ with the usual convention as to sign. The refractive index of the medium between surfaces λ and $\lambda+1$ being μ_λ, we have $\mu_0=\mu_n=1$ and $K_\lambda=R_\lambda(\mu_\lambda-\mu_{\lambda-1})$. It is convenient to suppose that the focal length of the thin system considered is unity, so that

$$K_1+K_2+K_3+ \ldots +K_n=1.$$

The chromatic condition is of the form

$$0=\delta K_1+\delta K_2+\delta K_3+ \ldots +\delta K_n$$
$$=(R_1-R_2)\delta\mu_1+(R_2-R_3)\delta\mu_2+ \ldots +(R_{n-1}-R_n)\delta\mu_{n-1}.$$

If $K_{1,\lambda}$ be written for $K_1+K_2+K_3+ \ldots +K_\lambda$ the values of A, B and ω may be calculated from

$$\left.\begin{aligned} A&=1+\Sigma R_\lambda\left(K_\lambda R_\lambda-2\frac{K^2_{1,\lambda}}{\mu_\lambda}+2\frac{K^2_{1,\lambda-1}}{\mu_{\lambda-1}}\right)\\ B&=A-1+\Sigma R_\lambda\left(\frac{K_{1,\lambda}}{\mu_\lambda}-\frac{K_{1,\lambda-1}}{\mu_{\lambda-1}}\right)\\ \varpi&=\Sigma\frac{K_\lambda}{\mu_{\lambda-1}\mu_\lambda}. \end{aligned}\right\} \quad \ldots (2),$$

where the summations are to be taken over all the surfaces of the system. The values of the other coefficients may be found at once from the relations (1). It should be particularly noted that ϖ is independent of the shape of any of the lenses, and is not altered by any rearrangement of them or by dividing up any lens to any extent, so long as the total power of the lenses made of each kind of glass remains the same. It may be expressed in the form

$$\varpi=\Sigma\left\{\frac{1}{\mu}\text{ (sum of powers of all lenses of refractive index }\mu)\right\}$$

where the summation is taken for all kinds of glass present in the system.

Uncemented Double Objective.

For constructive purposes the formulæ for the aberration coefficients may be conveniently arranged in another way. Let it be supposed that an objective is to be constructed of two kinds of glass of refractive indices μ and μ', and that the focal length is to be the same for two colours for which the differences of refractive index are $\delta\mu$ and $\delta\mu'$.

Put $R_1=R+r$, $R_2=r$, $R_3=r'$, $R_4=r'-R'$, $\mu_0=\mu_2=\mu_4=1$, $\mu_1=\mu$, $\mu_3=\mu'$, $K_1+K_2=K$, $K_3+K_4=K'$, $K+K'=1$.

Then $K=R(\mu-1)$, $K'=R'(\mu'-1)$ and the chromatic condition is $0=R\delta\mu+R'\delta\mu'$, and therefore

$$K=\frac{\nu}{\nu-\nu'},\quad K'=\frac{-\nu'}{\nu-\nu'},$$

where

$$\nu=\frac{\mu-1}{\delta\mu},\quad \nu'=\frac{\mu'-1}{\delta\mu'}.$$

Also $\varpi=\dfrac{K}{\mu}+\dfrac{K'}{\mu'}$, and all quantities are determined except r and r'. The two remaining aberration conditions are to be satisfied by a proper choice of these two quantities.

As there is frequently no a priori reason for placing the component lenses in one order rather than the reverse, it is convenient to put the formulæ in such a way that they can be as readily applied to the one case as to the other, and so far as possible avoid additional numerical work. The quantities to be expressed in terms of r and r' may therefore be C and

(B′−B). The reduction is easily made from formulæ (1) and (2), with the results,

$$\left.\begin{aligned} C=KR^2+K'R'^2+2KK'(R+R'-\varpi) \\ +Kr\left\{3R+2K'-\frac{2}{\mu}(K-K')\right\} \\ -K'r'\left\{3R'+2K+\frac{2}{\mu'}(K-K')\right\} \\ +Kr^2\left(1+\frac{2}{\mu}\right)+K'r'^2\left(1+\frac{2}{\mu'}\right) \end{aligned}\right\} \quad . \quad (3)$$

and

$$\left.\begin{aligned} B'-B=2\left\{KR-K'R'+Kr\left(1+\frac{1}{\mu}\right)\right. \\ \left.+K'r'\left(1+\frac{1}{\mu'}\right)\right\}-(K-K')\varpi \end{aligned}\right\}$$

By the second condition r' is expressed as a linear function of r, and by substituting this value in the first condition a quadratic equation is obtained for r. When this is solved the system is completely determined. As an illustration of the use of these formulæ let it be required to find the possible forms for a two lens astronomical telescope objective using the glasses for which Steinheil and Voit* carried out their investigations. The conditions to be satisfied are C=1, and when light approaches from the positive direction B′−B=2+ϖ, or from the negative direction B′−B=−(2+ϖ). The values to be assigned to

$$KR-K'R'+rK\left(1+\frac{1}{\mu}\right)+r'K'\left(1+\frac{1}{\mu'}\right)$$

are thus 1+Kϖ and −(1+K′ϖ) respectively.

The glasses employed are

$$\text{Crown}\begin{cases}\mu_D=1{\cdot}51806\\ \mu_V=1{\cdot}53356\end{cases} \qquad \text{Flint}\begin{cases}\mu_{D'}=1{\cdot}61358\\ \mu_{V'}=1{\cdot}64252\end{cases}$$

giving for the D line

$$K=2{\cdot}7348,\ K'=-1{\cdot}7348,\ R=5{\cdot}2789,\ R'=-2{\cdot}8273,$$

$$\varpi=\frac{K}{\mu}+\frac{K'}{\mu'}=1{\cdot}8015-1{\cdot}0751=0{\cdot}7264.$$

* "Handbuch der Angewandten Optik."

The condition C=1 gives

$$6{\cdot}3378r^2-3{\cdot}8851r'^2+17{\cdot}7177r+4{\cdot}3848r'+44{\cdot}9734=0.$$

B′−B=2+ϖ gives

$$4{\cdot}5363r-2{\cdot}8099r'+6{\cdot}5454=0,$$

and B′−B=−(2+ϖ) gives

$$4{\cdot}5363r-2{\cdot}8099r'+9{\cdot}2718=0.$$

The solutions are

$$\left.\begin{matrix} r=-3{\cdot}6410 \\ r'=-3{\cdot}5486 \end{matrix}\right\},\quad \left.\begin{matrix} r=+2{\cdot}4732 \\ r'=+6{\cdot}3221 \end{matrix}\right\},\quad \left.\begin{matrix} r=-5{\cdot}2443 \\ r'=-5{\cdot}1666 \end{matrix}\right\} \text{ and } \left.\begin{matrix} r=+0{\cdot}8630 \\ r'=+4{\cdot}6929 \end{matrix}\right\}$$

the first two for parallel light incident on the crown lens, and the last two for parallel light incident on the flint lens. These forms are illustrated in Fig. 1 on the same scale as Steinheil &

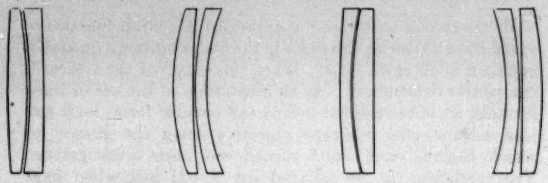

Fig. 1.

Voit's figures. Of these four forms the second and fourth are probably never used. It usually happens when alternative forms are available that the one involving the least curvatures is adopted, the probability being that with this construction a larger aperture can be used than with the other forms before the higher order aberrations become perceptible.

In the more desirable forms the curvatures of the inner surfaces are nearly, but not quite, equal to one another. This result is true of nearly all the glasses likely to be used in telescope construction. It is often a decided advantage to employ an objective in which these surfaces are of equal curvature and are cemented together. A very appreciable gain in the amount of light transmitted is secured by eliminating glass air surfaces in this way. In some cases it is not necessary to satisfy the assumed conditions to a high order of

accuracy, and mathematical exactitude in the removal of aberrations may be sacrificed to gain increased light transmitting qualities. It might be supposed, for example in the first solution obtained above, that the effect of slightly altering the shape of one or both of the lenses to make their curvatures alike, would be of little consequence, except perhaps in the case of a large objective, the difference in curvature being too small to be perceptible from the diagram. If we perform the calculations putting firstly $r=r'=-3{\cdot}6410$ and secondly $r=r'=-3{\cdot}5486$ we find that the aberration coefficients have the values $A=-3{\cdot}5052$, $B=-3{\cdot}2456$ and $A=-3{\cdot}4114$, $B=-2{\cdot}9923$ respectively. It will be observed that the signs of the errors introduced are the same, whether the crown lens is bent towards the flint or the flint towards the crown. An idea of the magnitude of the errors to which the above numbers correspond may be formed by comparing them with the values for a single thin lens of refractive index 1·5. If such a lens is planoconvex with the convex side towards the incident light, its coefficients will have the values $A=2\frac{1}{3}$, $B=2\frac{2}{3}$; if it is equiconvex $A=3\frac{1}{3}$, $B=2$; if planoconvex with the plane side towards the incident light $A=9$, $B=6$. Thus the spherical aberration introduced by making the combination cemented with the minimum change of shape is about equal, but opposite in sign, to that of the equiconvex lens, and the coma (measured by $A-B$) is approximately equal and opposite in sign to that of the planoconvex lens with the curved surface turned towards the incident light.

When central spherical aberration is present, rays incident parallel to the axis on the zone of diameter d will meet the axis at a distance $Ad^2/8f$ nearer to the lens than the principal focus for paraxial rays, where f is the focal length of the lens. The diameter of the so-called circle of confusion in the paraxial image plane is $Ad^3/8f^2$. Coma has been described as a zonal variation of the focal length, but this description is not strictly accurate, inasmuch as when it is present all rays from the same zone do not meet the image plane in one point. Rays from diametrically opposite points of any one zone meet the image plane in one and the same point, and the fact that both rays in an axial or primary plane do so agree is probably responsible for this way of expressing the meaning of coma. Rays in a primary plane meet the paraxial image plane in the points determined by assuming the focal length to be $f+3(A-B)d^2/8f$, and rays incident in a secondary plane as

though the focal length were $f+(A-B)d^2/8f$. Rays from other parts of the zone meet the image plane in the circle having the image points for rays in the primary and secondary planes as the extremities of a diameter. When coma and central spherical aberration are both present their effects on each ray are additive. In the lens taken as an illustration, if the aperture were one-tenth the focal length, the longitudinal spherical aberration introduced by modifying the lens to the extent necessary to enable the crown and flint glasses to be cemented together, would be nearly $\frac{1}{2}$ per cent. of the focal length, and the coma would correspond to a decrease in the focal length for the outer zones amounting to about one-eighth of 1 per cent. For many purposes such errors would be quite inadmissible. Before turning to systems which satisfy the conditions, and at the same time have only two glass air surfaces, we may note one or two cemented lenses of the type so far considered. With the above glasses, when there is no coma with the crown lens leading it is found that

$$r=r'=-3{\cdot}7914, \quad A=-3{\cdot}5684,$$

and with the flint lens leading

$$r=r'=-5{\cdot}3707, \quad A'=-2{\cdot}9842.$$

When spherical aberration is corrected the solutions are

$$\left.\begin{aligned} r=r'&=-5{\cdot}0080 \\ B&=-2{\cdot}1003 \end{aligned}\right\} \text{ and } \left\{\begin{aligned} r=r'&=-2{\cdot}5956 \\ B&=+2{\cdot}0645 \end{aligned}\right.$$

with the crown lens leading, and

$$\left.\begin{aligned} r=r'&=-6{\cdot}3242 \\ B'&=+1{\cdot}6462 \end{aligned}\right\} \text{ and } \left\{\begin{aligned} r=r'&=-4{\cdot}0947 \\ B'&=-2{\cdot}2027 \end{aligned}\right.$$

with the flint lens leading. Of these four solutions the second has the crown lens very approximately equiconvex, and the flint plano concave; this form is frequently adopted for small objectives on account of its small manufacturing cost.

Effect of Bending Thin Objectives.

When a lens system must have all its inner surfaces cemented together, only one degree of freedom is present apart from those obtainable by changes in the kinds of glass used, in the number

of component lenses and in the order in which they are arranged on the axis. In a thin lens this freedom is exercised by increasing the curvature of each refracting surface by the same amount. On substituting in equations (2) $r+R_\lambda$ for R_λ, $K_\lambda+r(\mu_\lambda-\mu_{\lambda-1})$ for K_λ, and $K_{1,\lambda}+r(\mu_\lambda-1)$ for $K_{1,\lambda}$, where r is

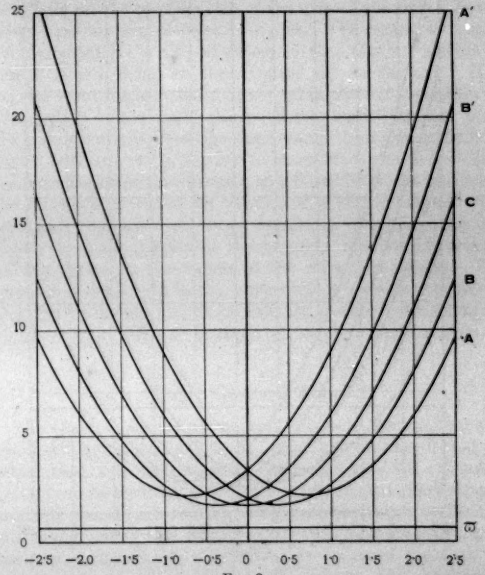

FIG. 2.

the increment of curvature of each surface of the system it will be found that

$$\left.\begin{aligned} C&=C_0+r(2B_0'-2B_0-\Sigma K_\lambda R_\lambda)+r^2(1+2\varpi).\\ \text{and}\quad B'-B&=B_0'-B_0+2r(1+\varpi)\end{aligned}\right\}, \qquad (4)$$

where the suffix $_0$ is attached to the values of the aberration coefficients for $r=0$. Thus when these coefficients are known for any thin system, the effect of any bending of the system as a whole is determinable at once when $\Sigma K_\lambda R_\lambda$ has been found

for the unbent system. Each product contributing to this sum has already been evaluated in the course of finding the aberration coefficients by equations (2).

The standard form for the equations expressing the effect of

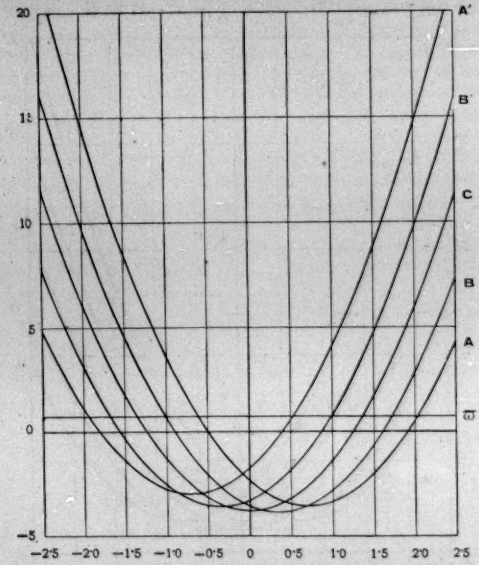

Fig. 3.

bending may be taken as that for which $r=0$ when C is a minimum. The equations then become

$$\begin{aligned}
A &= A_0 - 2r(1+\varpi) + r^2(1+2\varpi) \\
B &= B_0 - r(1+\varpi) + r^2(1+2\varpi) \\
C &= C_0 + r^2(1+2\varpi) \\
B' &= B_0' + r(1+\varpi) + r^2(1+2\varpi) \\
A' &= A_0' + 2r(1+\varpi) + r^2(1+2\varpi)
\end{aligned}$$

The curves expressing the variation of the coefficients with r as independent variable, are thus equal parabolas with their

axes vertical and separated horizontally at equal intervals of $(1+2\varpi)/2(1+\varpi)$. Fig. 2 shows these curves for a single lens of refractive index 1·5. In Fig. 3 the curves for cemented doublets made of the glasses used in the foregoing numerical illustrations are given. In the case of the single lens the curves lie wholly on the positive side of the axis of abscissæ. In the other case the curves cross this axis. The cemented lens is overcorrected for a distant object in that the intersection of curves A and B lies on the negative side of the axis. It is evident from the fact that a cemented doublet is overcorrected when made of the usual glasses, but is undercorrected when the refractive index has the same value throughout the lens, that it is theoretically possible to choose two glasses such that the intersection of the curves A and B will lie on the axis when the system is corrected for colour and is also cemented. The practical realisation of these conditions depends upon our ability to make glasses of the required relative dispersions, having regard to the values of the refractive indices. This question is dealt with later; meanwhile it is worth considering how the conditions can be fulfilled for a cemented achromatic objective when it has to be made of two given kinds of glass.

Triple Cemented Lenses.

The simplest way of attempting this is to assume that two lenses will be made of one kind of glass, and that the lens of the other kind will be cemented between these two. Such a system can be regarded as built up of two cemented achromatic doublets placed back to back. Now the aberration coefficients of a compound system composed of two similar systems placed back to back are easily expressed in terms of those of the components. Let large capitals denote the aberration coefficients of the compound system, and small capitals those of the simple systems. If the compound system of power unity be made up of simple systems of powers $1-\theta$ and θ, the relations

$$\mathrm{C}=\mathrm{C}+\theta(1-\theta)(\mathrm{A}-4\mathrm{B}-1)$$

and

$$\mathrm{B}'-\mathrm{B}=(1-2\theta)(\mathrm{A}-4\mathrm{B}-1+3\mathrm{C}+\varpi)$$

hold whether the simple systems are thin or not. When the

simple systems are thin these relations reduce, in virtue of equations (1) to

$$\left.\begin{aligned} \mathbf{C} &= \mathrm{C}-\theta(1-\theta)(2\mathrm{B}+\mathrm{C}) = \mathrm{C}-\theta(1-\theta)(3\mathrm{C}+\varpi-\mathrm{B}'+\mathrm{B}) \\ \text{and} \\ \mathbf{B}'-\mathbf{B} &= (1-2\theta)(\mathrm{B}'-\mathrm{B}) \end{aligned}\right\} \quad (5)^*$$

Let the value of $\Sigma \mathrm{K}_\lambda \mathrm{R}_\lambda$ for the simple system be denoted by σ; then for the compound system $\Sigma \mathrm{K}_\lambda \mathrm{R}_\lambda = (1-2\theta)\sigma$. Combining with equations (5) the resulting change in the coefficients due to a change of curvature r in each surface from equations (4) we obtain the generalised formulæ

$$\left.\begin{aligned} \mathbf{C} &= \mathrm{C}-\theta(1-\theta)(2\mathrm{B}+\mathrm{C})+r(1-2\theta)(2\mathrm{B}'-2\mathrm{B}-\sigma)+r^2(1+2\varpi) \\ \mathbf{B}'-\mathbf{B} &= (1-2\theta)(\mathrm{B}'-\mathrm{B})+2r(1+\varpi) \end{aligned}\right\} \quad (6)$$

giving the effect of combining two systems together and bending the combined system to an arbitrary extent. In order that the coefficients may be those for a system with only two glass air surfaces it is only necessary that the sub-systems, in addition to having their inner surfaces cemented together, should have their last surfaces plane in the form to which the values of C, B′−B, and σ used on the right of equations (6) refer.

The form of these equations shows that the general problem of satisfying two aberration conditions is resolved into the solution of a quadratic equation. Thus—apart from the occurrence of imaginary roots—four thin cemented achromatic triplet objectives can be made with two given kinds of glass to satisfy two arbitrary aberration conditions, two of them having one kind of glass enclosed between lenses of the other kind, and two with this glass enclosing a lens of the second type of glass. Since the triplet form provides solutions of the general problem we have been considering there is nothing to be gained by the consideration of thin systems of a more complex type. With the Steinheil & Voit glasses, it is readily found that a double

* These results may of course be verified by substituting in formulæ (2) for the powers, curvatures and refractive indices the following series of values :—

$$(1-\theta)\mathrm{K}_1, (1-\theta)\mathrm{K}_2, (1-\theta)\mathrm{K}_3, \ldots (1-\theta)\mathrm{K}_n, \theta\mathrm{K}_n, \theta\mathrm{K}_{n-1}, \theta\mathrm{K}_{n-2}, \ldots \theta\mathrm{K}_2, \theta\mathrm{K}_1$$

$$(1-\theta)\mathrm{R}_1, (1-\theta)\mathrm{R}_2, (1-\theta)\mathrm{R}_3, \ldots (1-\theta)\mathrm{R}_n, -\theta\mathrm{R}_n, -\theta\mathrm{R}_{n-1}, -\theta\mathrm{R}_{n-2}, \ldots -\theta\mathrm{R}_2, -\theta\mathrm{R}_1$$

$$1, \mu_1, \mu_2, \mu_3, \ldots \mu_{n-1}, 1, \mu_{n-1}, \mu_{n-2}, \ldots \mu_2, \mu_1, 1.$$

cemented objective with the second surface of the flint lens plane has

$$C=3{\cdot}0890,\ B'-B=6{\cdot}0551,\ \Sigma K_\lambda R_\lambda=3{\cdot}8773,$$

and therefore for a cemented triplet

$$C=2{\cdot}4528r^2+8{\cdot}2329r(1-2\theta)-3{\cdot}9383\theta(1-\theta)+3{\cdot}0890$$

$$B'-B=3{\cdot}4528r+6{\cdot}0551(1-2\theta),$$

the solutions of which when $C=1$ and $B'-B=2+\varpi$ are

$$\left.\begin{matrix}\theta=0{\cdot}1783\\ r=-0{\cdot}3385\end{matrix}\right\} \text{ and } \left.\begin{matrix}\theta=0{\cdot}8464\\ r=2{\cdot}0045\end{matrix}\right\},$$

giving for the curvatures of the four surfaces

$$\left.\begin{matrix}1{\cdot}6758\\ -2{\cdot}6616\\ 0{\cdot}1657\\ -0{\cdot}7758\end{matrix}\right\} \text{ and } \left.\begin{matrix}2{\cdot}3811\\ 1{\cdot}5701\\ 4{\cdot}3974\\ -0{\cdot}0705\end{matrix}\right\} \text{ respectively.}$$

In the doublet with flint lens leading and last surface plane $C=-2{\cdot}3528$, $B'-B=2{\cdot}4097$, $\Sigma K_\lambda R_\lambda=1{\cdot}0259$, leading to equations for this form of cemented triplet

$$C=2{\cdot}4528r^2+3{\cdot}7935r(1-2\theta)+8{\cdot}7417\theta(1-\theta)-2{\cdot}3528$$

$$B'-B=3{\cdot}4528r+2{\cdot}4097(1-2\theta)$$

and to the solutions

$$\left.\begin{matrix}\theta=0{\cdot}6389\\ r=0{\cdot}9835\end{matrix}\right\} \text{ and } \left.\begin{matrix}\theta=0{\cdot}3210\\ r=0{\cdot}5397\end{matrix}\right\},$$

with curvatures

$$\left.\begin{matrix}1{\cdot}8688\\ 2{\cdot}8897\\ -2{\cdot}3892\\ -0{\cdot}5828\end{matrix}\right\} \text{ and } \left.\begin{matrix}2{\cdot}2044\\ 4{\cdot}1244\\ -1{\cdot}1546\\ -0{\cdot}2471\end{matrix}\right\} \text{ respectively.}$$

As was to be expected the curvatures compare favourably with those for the double forms of objective satisfying the same conditions, viz. :—

$$\left.\begin{matrix}1{\cdot}6379\\ -3{\cdot}6410\\ -3{\cdot}5486\\ -0{\cdot}7213\end{matrix}\right\}, \left.\begin{matrix}7{\cdot}7521\\ 2{\cdot}4732\\ 6{\cdot}3221\\ 9{\cdot}1494\end{matrix}\right\}, \left.\begin{matrix}2{\cdot}3393\\ 5{\cdot}1666\\ 5{\cdot}2443\\ -0{\cdot}0346\end{matrix}\right\}, \left.\begin{matrix}-7{\cdot}5202\\ -4{\cdot}6929\\ -0{\cdot}8630\\ -4{\cdot}4159\end{matrix}\right\}.$$

It will be noted that one of the forms with crown glass outside is slightly better than the more favourable form with flint outside, as regards smallness of curvature, and that the

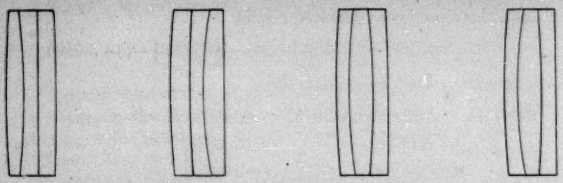

Fig. 4.

other two forms are not as satisfactory in this respect as the best doublet form. The four triplets are illustrated in Fig. 4.

Cemented Double Objectives for Telescopes.*

From the point of view of reduced cost of manufacture a doublet lens is preferable to a triplet, and it is of interest to see what properties the glasses to be used must have if a cemented objective is to be free from colour and from spherical aberration and coma. Assuming a refractive index for the flint lens of 1·6, the ratio of the dispersions of crown and flint glasses for various refractive indices of the crown lens is shown by the curve of Fig. 5. This ratio demands less dispersive power from the crown lens than is found in the ordinary glasses ; even when the ν for the flint lens is as low as is obtainable with a refractive index of 1·6, the only glasses which satisfy the conditions are phosphate crowns, which, on account of their instability, are out of the question, and medium baryta crowns

* Cemented doublets free from spherical aberration and coma for a distant object have been very thoroughly investigated by Harting, who has prepared tables giving the necessary curvatures for the surfaces and the relative dispersions of the glasses for given refractive indices, and has further considered what glass combinations satisfy the conditions. His list of possible combinations does not, of course, include the fluor crown combination mentioned above, as this glass has been introduced since Hartings' investigation was carried out, and his tables are not sufficiently extended to include such glasses. Von Hoegh has given equations for the determination of such doublets when the object is at a given finite distance from the lens. The diagrams given above were obtained by a method which avoids the solution of a quintic equation. An account of the work of both Harting and Von Hoegh on this subject will be found in Gleichen's "Lehrbuch der Geometrischen Optik," p. 323.

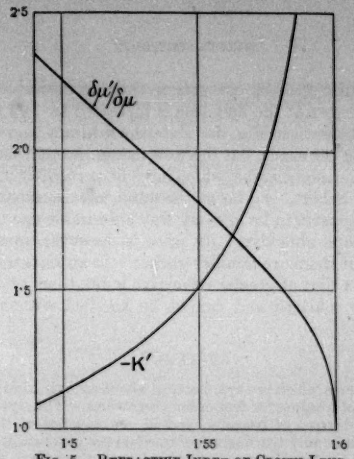

FIG. 5.—REFRACTIVE INDEX OF CROWN LENS.

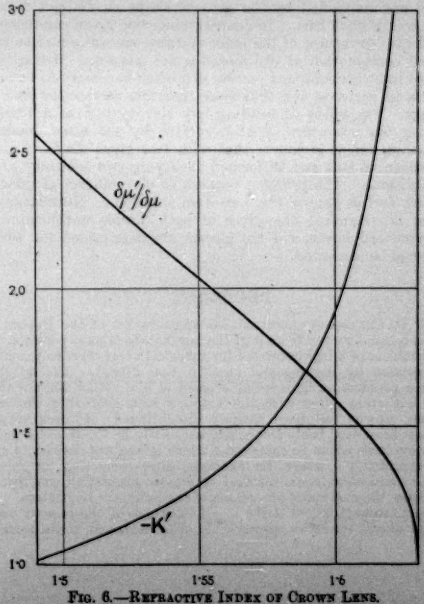

FIG. 6.—REFRACTIVE INDEX OF CROWN LENS.

which require greater curvatures than is desirable. If the refractive index of the flint lens is increased to 1·63 the curve obtained is shown in Fig. 6. Here the medium baryta crowns are again on the curve, but this now passes through the position of one of the fluor crowns, which have been recently introduced by Messrs. Schott. So far as the published descriptions state, this glass appears to be in every way a desirable one to employ in a telescope objective. Its price is, however, much higher than that of the more familiar glasses. It appears undesirable to employ a flint of greater refractive index than 1·63, as such glasses are coloured and cannot be handled without risk of tarnishing.

ABSTRACT.

Lens systems which are symmetrical about an axis have in general six degrees of freedom for first order aberrations. Thin systems have only three degrees of freedom, and in consequence of the limited range of glasses only two degrees of freedom are practically available. In achromatic combinations of two lenses these two degrees of freedom are controlled by the general shape as distinct from the total power of each lens. In general when two given conditions are satisfied the curvatures of the inner surfaces are not equal, so that a cemented combination of two lenses is not possible. Owing to the increased light transmitting powers it is often necessary to have only two glass air surfaces, and thus more than two component lenses are necessary. The effect of bending any thin system as a whole by increasing the curvature of each surface by the same amount is investigated, and it is shown that with two given kinds of glass a triple cemented lens can be formed satisfying two arbitrary aberration conditions. The problem reduces to the solution of quadratic equations, and in general there are four solutions. Illustrations are given of astronomical objectives of both double uncemented and triple cemented forms, and the glasses are determined for which a doublet can be cemented.

DISCUSSION.

Mr. S. D. Chalmers expressed his appreciation of the Paper. One could get a rapid survey from it of the results which were possible. The special problem of a lens corrected for spherical aberration and coma had been discussed by Harting and also by Von Hoegh. The latter had solved the problem on lines which seemed to him rather simpler to use. He did not quite agree with the author's statement that the use of phosphate crowns had been entirely discontinued. Glasses very like these have been used fairly frequently, especially by Continental makers. There were cases where in computing a lens it was not desired to achromatise completely; where, for example, after correcting for spherical aberration and coma, some residual chromatic aberration was desirable to neutralise the chromatic aberration of the prisms in binoculars.

Mr. A. Campbell asked if the small thickness of the cement used in cemented lenses made an appreciable difference, and if its refractive

properties entered into the calculations. What was the range of refractive index available in the modern optical glasses ?

The AUTHOR, in reply, was surprised to hear that phosphate glasses were still employed in any case in which they could possibly be avoided. The cement used was always Canada balsam and was entirely neglected in all computations. A range of refractive index of 1·47 to 1·79 was available, but the extreme values were rarely used, 1·49 to about 1·65 or so being the range usually utilised.

XXXVII.—*On Tracing Rays Through an Optical System.* By T. SMITH, *B.A.* (*From the National Physical Laboratory.*)

RECEIVED MAY 27, 1915.

THE most troublesome calculations which have to be made in computing an optical system are those relating to rays not lying in an axial plane. The methods hitherto used are trigonometrical. Perhaps the formulæ most extensively employed are those of Von Seidel. A complete account of them will be found in an appendix to Steinheil & Voit's "Handbuch der Angewandten Optik." They may, with minor changes in the notation, be summarised as follows.

Let OP and O′P be the incident and refracted rays for a

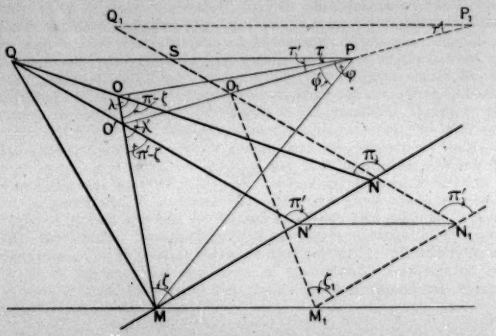

DIAGRAM ILLUSTRATING VON SEIDEL'S METHOD OF TRACING A GENERAL RAY THROUGH A LENS SYSTEM.

spherical surface whose centre is at M. Take O and O′ in the plane through M normal to the axis of the system, and let Q be the foot of the perpendicular let fall from the point of refraction P to this plane. In the diagram, lines in this plane are drawn as continuous thick lines, and lines not in a plane normal to the axis are shown thin. The interrupted lines refer to refraction at a subsequent surface. The angles of incidence and emergence are φ and φ'; the angles made by the incident and refracted rays with the axis are τ, τ'. The

angles made by QO and QO′ with an arbitrary direction MN′N in the normal plane are π and π'. The points M, O′ and O lie on a straight line inclined at an angle ζ with MN, so that the angles MON and MO′N′ are $\pi-\zeta$ and $\pi'-\zeta$ respectively. Let the angles MOP and MO′P be called λ and λ', the lengths MO and MO′ U and U′, and the radius of the sphere R. Then the equation of refraction is

$$\mu' \sin \varphi' = \mu \sin \varphi,$$

and from the figure the relations,

$$\cos \lambda = \sin \tau \cos (\pi - \zeta),$$

$$\sin \varphi = \frac{\mathrm{U} \sin \lambda}{\mathrm{R}},$$

$$\lambda' = \lambda + \varphi - \varphi',$$

$$\mathrm{U}' = \frac{\mathrm{R} \sin \varphi'}{\sin \lambda'} = \mathrm{U} \frac{\mu \sin \lambda}{\mu' \sin \lambda'},$$

$$\frac{\sin \tau' \sin (\pi' - \zeta)}{\sin \lambda'} = \frac{\sin \tau \sin (\pi - \zeta)}{\sin \lambda},$$

$$\sin \tau' \cos (\pi' - \zeta) = \cos \lambda'$$

are found.

Let the refracted ray meet the next surface whose centre is M_1 in P_1; and let the notation applied to the previous surface hold throughout for this surface with the addition of the suffix 1. The plane containing PQ and $\mathrm{P}_1\mathrm{Q}_1$ is parallel to the axis, and $\mathrm{Q}_1\mathrm{O}_1$ meets PQ in a point S in this plane. The intersection of this plane with the plane $\mathrm{MM}_1\mathrm{N}'$ is the line $\mathrm{N}'\mathrm{N}_1$, and this is parallel to the axis. Thus $\pi_1 = \pi'$, $\mathrm{QN}' = \mathrm{SN}_1$, and the distances of QN′ and SN_1 from the axis are equal, leading to

$$\mathrm{U}_1 \sin (\pi' - \zeta_1) = \mathrm{U}' \sin (\pi_1 - \zeta),$$

and

$$\mathrm{U}_1 \cos (\pi' - \zeta_1) = \mathrm{U}' \cos (\pi' - \zeta) - c \tan \tau',$$

where $c = \mathrm{MM}_1$. The four co-ordinates U, ζ, τ, π are selected to define the incident ray at one surface, and the above equations determine the corresponding quantities for the next surface, since the equation $\tau_1 = \tau'$ follows by definition.

These equations have the advantage of being in a form suitable for logarithmic computation, but the process is very tedious—nine equations have to be solved for each surface—and the method does not readily indicate what modifications should be made in the system when the ray does not emerge as is desired. An algebraic method is preferable precisely

because it is free from this objection, and the fact that it is not as well suited for logarithmic work is of no importance where a calculating machine is employed.

The algebraic formulæ to be used should, if possible, be comparable with those used for calculating paraxial rays. Let the system consist of n refracting surfaces, distinguished by the numbers 1, 2, 3, . . . λ . . . n, the radii of curvature being r_1, r_2, . . . r_λ, . . . r_n reckoned positive when the surfaces are convex to incident light passing from surface 1 to surface n. The refractive index of the medium between surfaces λ and $\lambda+1$ will be denoted by μ_λ, and the axial separation of these surfaces by t_λ. The power of the surface λ is $\kappa_\lambda=\dfrac{\mu_\lambda-\mu_{\lambda-1}}{r_\lambda}$, the sign being that commonly used by opticians. The power of the complete system and other constants are found by successive applications of the formulæ

$$\kappa_{1,\lambda}=\kappa_{1,\lambda-1}+\kappa_\lambda\frac{\partial\kappa_{1,\lambda}}{\partial\kappa_\lambda}, \qquad \kappa_{\lambda,n}=\kappa_{\lambda+1,n}+\kappa_\lambda\frac{\partial\kappa_{\lambda,n}}{\partial\kappa_\lambda},$$

$$\frac{\partial\kappa_{1,\lambda}}{\partial\kappa_\lambda}=\frac{\partial\kappa_{1,\lambda-1}}{\partial\kappa_{\lambda-1}}-\frac{t_{\lambda-1}}{\mu_{\lambda-1}}\kappa_{1,\lambda-1}, \qquad \frac{\partial\kappa_{\lambda,n}}{\partial\kappa_\lambda}=\frac{\partial\kappa_{\lambda+1,n}}{\partial\kappa_{\lambda+1}}-\frac{t_\lambda}{\mu_\lambda}\kappa_{\lambda+1,n},$$

where $\kappa_{1,\lambda}$ is the power of the system composed of the surfaces 1, 2, 3, . . . λ.

NOTE.—The advantage of using the second set of formulæ given above, instead of those which give $\dfrac{\partial\kappa_{1,n}}{\partial\kappa_1}$ and $\dfrac{\partial^2\kappa_{1,n}}{\partial\kappa_1\partial\kappa_n}$ directly is that in numerical calculations, which from their nature are approximations, the equality of the two values found for $\kappa_{1,n}$ is a better check on the accuracy of the result obtained than the satisfaction to the same number of significant figures of the identity

$$\frac{\partial\kappa_{1,n}}{\partial\kappa_1}\cdot\frac{\partial\kappa_{1,n}}{\partial\kappa_n}-\kappa_{1,n}\frac{\partial^2\kappa_{1,n}}{\partial\kappa_1\partial\kappa_n}=1.$$

If, now, a paraxial ray enters the system inclined at an angle θ_0 with the axis and meets the first surface at a distance S_1 from the axis, the emergent angle θ_n and the distance S_n at which the last surface is met, are given by the equations

$$\mu_n\theta_n=\mu_0\theta_0\frac{\partial\kappa_{1,n}}{\partial\kappa_1}-S_1\kappa_{1,n},$$

$$S_n=-\mu_0\theta_0\frac{\partial^2\kappa_{1,n}}{\partial\kappa_1\partial\kappa_n}+S_1\frac{\partial\kappa_{1,n}}{\partial\kappa_n};$$

or as an alternative to the latter,

$$\mu_0\theta_0=\mu_n\theta_n\frac{\partial\kappa_{1,n}}{\partial\kappa_n}+S_n\kappa_{1,n}.$$

Turning, now, to the general case, let a ray passing between surfaces λ and $\lambda+1$ have direction cosines L_λ, M_λ, N_λ referred to rectangular axes of which the axis of x coincides with the optical axis, and let d_λ be the length of that part of the ray which lies between these two surfaces, which must be symmetrical about the optical axis, but need not for the moment be assumed spherical. The direction cosines of the normal to the λth surface at the point of refraction will be denoted by l_λ, m_λ, n_λ, and the length of the normal intercepted between the optical axis and the surface by r_λ, so that r_λ is the transverse radius of curvature of the surface at the point of refraction.

The equations of refraction are

$$\frac{\mu_\lambda L_\lambda - \mu_{\lambda-1} L_{\lambda-1}}{l_\lambda} = \frac{\mu_\lambda M_\lambda - \mu_{\lambda-1} M_{\lambda-1}}{m_\lambda} = \frac{\mu_\lambda N_\lambda - \mu_{\lambda-1} N_{\lambda-1}}{n_\lambda}. \quad . \quad (1)$$

Denote each of these equal quantities by $r_\lambda K_\lambda$; K_λ is defined as the power of the surface for this particular ray. If D_λ is the deviation suffered by the ray the way in which K_λ varies with the obliquity of the incidence is indicated by the equation,

$$r_\lambda^2 K_\lambda^2 = \mu_\lambda^2 + \mu_{\lambda-1}^2 - 2\mu_\lambda \mu_{\lambda-1} \cos D_\lambda.$$

The equations giving d_λ are

$$d_\lambda = \frac{a_\lambda/\mu_\lambda + l_\lambda r_\lambda - l_{\lambda+1} r_{\lambda+1}}{L_\lambda} = \frac{m_\lambda r_\lambda - m_{\lambda+1} r_{\lambda+1}}{M_\lambda} = \frac{n_\lambda r_\lambda - n_{\lambda+1} r_{\lambda+1}}{N_\lambda}, \quad (2)$$

where a_λ/μ_λ is the intercept on the axis between normals to surfaces λ and $\lambda+1$ at the points where they are met by the ray. From equations (1) and (2) it is seen that if $K_{1,n}$, &c., are defined by the formulæ,

$$\left.\begin{aligned} K_{1,\lambda} &= K_{1,\lambda-1} + K_\lambda \frac{\partial K_{1,\lambda}}{\partial K_\lambda}, \\ \frac{\partial K_{1,\lambda}}{\partial K_\lambda} &= \frac{\partial K_{1,\lambda-1}}{\partial K_{\lambda-1}} - \frac{d_{\lambda-1}}{\mu_{\lambda-1}} K_{1,\lambda-1}, \\ &\&c., \end{aligned}\right\} \quad . \quad . \quad . \quad (3)$$

the exact equations for refraction through the system are

$$\left.\begin{aligned} \mu_n M_n &= \mu_0 M_0 \frac{\partial K_{1,n}}{\partial K_1} + m_1 r_1 K_{1,n}, \\ \mu_n N_n &= \mu_0 N_0 \frac{\partial K_{1,n}}{\partial K_1} + n_1 r_1 K_{1,n}, \\ m_n r_n &= \mu_0 M_0 \frac{\partial^2 K_{1,n}}{\partial K_1 \partial K_n} + m_1 r_1 \frac{\partial K_{1,n}}{\partial K_n}, \\ n_n r_n &= \mu_0 N_0 \frac{\partial^2 K_{1,n}}{\partial K_1 \partial K_n} + n_1 r_1 \frac{\partial K_{1,n}}{\partial K_n}, \end{aligned}\right\} \quad . \quad . \quad (4)$$

or, as alternative forms to the two latter equations,

$$\mu_0 M_0 = \mu_n M_n \frac{\partial K_{1,n}}{\partial K_n} - m_n r_n K_{1,n},$$

$$\mu_0 N_0 = \mu_n N_n \frac{\partial K_{1,n}}{\partial K_n} - n_n r_n K_{1,n}$$

may be used.

If, then, sufficiently simple formulæ can be obtained for the determination of the K's and d's, all rays can be traced through an axial system employing exactly the same formulæ as are used for paraxial rays. The required expressions evidently involve the shape of the section of the refracting surfaces by an axial plane. The most important case is that in which all the surfaces are spherical, so that $\frac{a_\lambda}{\mu_\lambda}$ is the distance between the centres of curvature of surfaces λ and $\lambda+1$. Let the lengths of the perpendiculars let fall on to the path of the ray in medium μ_λ from the centres of curvature of these two surfaces be $\frac{h_\lambda}{\mu_\lambda}$ and $\frac{h_{\lambda+1}}{\mu_\lambda}$ respectively. Then

$$\mu_\lambda d_\lambda = L_\lambda a_\lambda + \sqrt{\mu_\lambda^2 r_\lambda^2 - h_\lambda^2} - \sqrt{\mu_\lambda^2 r_{\lambda+1}^2 - h_{\lambda+1}^2} \quad . \quad (5)$$

Also from the law of refraction the length of the perpendicular from the centre of curvature of surface λ to the incident ray is $\frac{h_\lambda}{\mu_{\lambda-1}}$, that is to say h_λ is unaltered by refraction.

It follows from the definition given of K_λ that

$$r_\lambda^2 K_\lambda = \sqrt{\mu_\lambda^2 r_\lambda^2 - h_\lambda^2} - \sqrt{\mu_{\lambda-1}^2 r_\lambda^2 - h_\lambda^2}. \quad . \quad . \quad . \quad (6)$$

Now let $l_{\lambda+1}$, $m_{\lambda+1}$, $n_{\lambda+1}$ be eliminated from equations (2), using the relation $l_{\lambda+1}^2 + m_{\lambda+1}^2 + n_{\lambda+1}^2 = 1$; and let d_λ be given its value from equation (5). The result is

$$h_{\lambda+1}^2 = h_\lambda^2 - 2a_\lambda L_\lambda \sqrt{\mu_\lambda^2 r_\lambda^2 - h_\lambda^2} + 2a_\lambda \mu_\lambda r_\lambda l_\lambda + a_\lambda^2 (1 - L_\lambda^2). \quad (7)$$

Equations (5), (6) and (7), together with

$$\mu_\lambda L_\lambda = \mu_{\lambda-1} L_{\lambda-1} + l_\lambda r_\lambda K_\lambda$$

and

$$l_\lambda r_\lambda = \frac{a_{\lambda-1}}{\mu_{\lambda-1}} + l_{\lambda-1} r_{\lambda-1} - L_{\lambda-1} d_{\lambda-1},$$

determine all the K's and d's of the system as was required. Since l_n and L_n have been obtained in the course of this preliminary work, the final result may be checked by verifying the equations

$$l_n^2+m_n^2+n_n^2=L_n^2+M_n^2+N_n^2=1$$

$$h_n^2=\mu_n^2r_n^2\{1-(L_nl_n+M_nm_n+N_nn_n)^2\}$$

and

$$\mu_nr_n\{M_nn_n-N_nm_n\}=\mu_0r_1\{M_0n_1-N_0m_1\}$$

$$=\frac{\mu_0\mu_n(M_0N_n-N_0M_n)}{K_{1,n}}.$$

When the performance of the system is unsatisfactory, it appears probable that a comparison of the K's and d's with the corresponding κ's and t's will suggest what modifications are necessary to amend the system in the direction desired.

It is of interest to notice some deductions from equations (4). Suppose x, y, z are the co-ordinates of a point on the incident ray and ξ, η, ζ those of a point on the refracted ray, the origins of both systems of co-ordinates being on the axis. If μ_0P_0 is the length of the incident ray between x, y, z and the first surface, and μ_nP_n the length of the emergent ray between ξ, η, ζ and the last surface,

$$\left.\begin{aligned} y&=-m_1r_1-\mu_0M_0P_0\\ z&=-n_1r_1-\mu_0N_0P_0\\ \eta&=-m_nr_n+\mu_nM_nP_n\\ \zeta&=-n_nr_n+\mu_nN_nP_n \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

or from equations (4)

$$K_{1,n}\eta=-\mu_0M_0\left\{\left(P_0K_{1,n}-\frac{\partial K_{1,n}}{\partial K_1}\right)\left(P_nK_{1,n}-\frac{\partial K_{1,n}}{\partial K_n}\right)-1\right\}$$

$$-K_{1,n}y\left(P_nK_{1,n}-\frac{\partial K_{1,n}}{\partial K_n}\right)$$

$$K_{1,n}\zeta=-\mu_0N_0\left\{\left(P_0K_{1,n}-\frac{\partial K_{1,n}}{\partial K_1}\right)\left(P_nK_{1,n}-\frac{\partial K_{1,n}}{\partial K_n}\right)-1\right\}$$

$$-K_{1,n}z\left(P_nK_{1,n}-\frac{\partial K_{1,n}}{\partial K_n}\right).$$

Now let ξ, η, ζ be the point on the refracted ray which lies in the same axial plane as x, y, z. Then $\frac{\eta}{y}=\frac{\zeta}{z}=G$, say.

This requires

$$\left.\begin{aligned} P_0K_{1,n}&=\frac{\partial K_{1,n}}{\partial K_1}-\frac{1}{G}\\ P_nK_{1,n}&=\frac{\partial K_{1,n}}{\partial K_n}-G \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (9)$$

formulæ exactly corresponding to those for the determination of pairs of conjugate foci on the axis. If, then, conjugate points on a general ray are defined as the intersections of the incident and emergent portions of any ray with any plane through the axis of the system, all laws relating to conjugate foci on the axis will also hold of conjugate points on any ray; instead of transverse magnification, the ratio of the distances of the two points from the axis is to be taken; and instead of principal foci, principal points for the ray which are conjugate to the points at infinity on the ray.

It at once follows that the image of a plane can only be free from curvature for more than one magnification if the rays before and after refraction are equally inclined to the optical axis. For assuming that for magnification G_0 the object and image are plane, when the object has been moved along the axis through a distance $\frac{1}{\kappa_{1,n}}\left(\frac{1}{G}-\frac{1}{G_0}\right)$, the object point on the ray considered will have been moved through a distance $\frac{1}{L_0\kappa_{1,n}}\left(\frac{1}{G}-\frac{1}{G_0}\right)$, and since the new magnification is to be G, this must be equal to $\frac{1}{K_{1,n}}\left(\frac{1}{G}-\frac{1}{G_0}\right)$, or $K_{1,n}=L_0\kappa_{1,n}$.

Similarly, the movement of the image plane is $\frac{1}{\kappa_{1,n}}(G-G_0)$ along the axis and $\frac{1}{L_n\kappa_{1,n}}(G-G_0)$ or $\frac{1}{K_{1,n}}(G-G_0)$ along the ray. Thus $K_{1,n}=L_n\kappa_{1,n}$ must be satisfied if there is no curvature for the new magnification.

From the two values of $K_{1,n}/\kappa_{1,n}$ it is seen that L_n must be equal to L_0, the result stated.

If m_1, n_1, &c., are eliminated from equations (8) by means of

(4), and P_0 and P_n are given the values indicated in (9), the results are

$$y=\frac{1}{GK_{1,n}}(\mu_0 M_0-G\mu_n M_n),$$

$$z=\frac{1}{GK_{1,n}}(\mu_0 N_0-G\mu_n N_n),$$

$$\eta=\frac{1}{K_{1,}}(\mu_0 M_0-G\mu_n M_n),$$

$$\xi=\frac{1}{K_{1\,n}}(\mu_0 N_0-G\mu_n N_n),$$

that is to say, the co-ordinates of any pair of conjugate points satisfy the relations

$$K_{1,n}=\frac{\mu_0 M_0}{\eta}-\frac{\mu_n M_n}{y}=\frac{\mu_0 N_0}{\zeta}-\frac{\mu_n N_n}{z}. \quad . \quad . \quad . \quad . \quad (10)$$

Equations (10) may be taken as a basis for the definition of conjugate points, in which case the definition holds when the ray considered lies in an axial plane. For such rays the one conjugate point marks the intersection of the ray with the radial focal line formed by rays passing through the other point. This follows from simple considerations of continuity: assuming the existence of such a focal line its co-ordinates η, ζ are given by (10) for a ray lying very close to the axial plane but not in it; and the values of $K_{1,n}$, M_0, M_n, N_0, N_n may be made to differ by less than an arbitrarily assigned small quantity from those for the ray actually in the axial plane. The assumption of the existence of the focal line is readily justified, for from symmetry the path of the ray in the axial plane has either a maximum or a minimum value.

ABSTRACT.

Trigonometrical formulæ have been used for tracing rays not lying entirely in one plane through optical systems, as these can readily be arranged in a form suitable for logarithmic calculation. When a calculating machine is available such computations can be carried out more expeditiously by using algebraic formulæ given in the Paper; in form these exactly correspond with the expressions for paraxial rays, and a comparison of the numerical result appears likely to suggest what alterations should be made when a general ray does not behave as desired. It is pointed out that if the two points in which a general ray meets an axial plane are defined as conjugate points, all pairs of conjugate points on a ray are connected by the same relations as hold for object and image points for paraxial rays,

and the theory for paraxial rays can be extended to rays in genera by placing a suitable interpretation on magnification, &c. The definition of conjugate points can be extended to include rays lying in axial planes, in which case the one point marks the intersection of the ray with the radial focal line formed by rays passing through its conjugate.

DISCUSSION.

Mr. S. D. CHALMERS was particularly interested in the method of tracing rays through a system. The relationships between the various quantities involved was of considerable interest. The author's methods of treatment seem to present many possibilities in addition to the cases actually quoted. By extending them he may simplify very considerably many important computations. Some time ago he had used somewhat similar methods to express the image position of an object in terms of the separations of the surfaces and the height of the intercepts of a standard ray on the different surfaces.

Mr. T. SMITH, in reply, agreed that if one had time to go into all the cases which might arise many interesting and useful results would follow. It was easy from the formulæ he had quoted to obtain an idea of what other general relations are likely to exist in an optical system, and to extend the meaning to be attributed to various conditions, such, for example, as the sine condition of Abbé. He hoped to treat more fully of these matters later.

XXXVIII. *On an Investigation of the Accuracy of the Lens and Drop Method of Measuring Refractive Index.* *By* H. Redmayne Nettleton, *B.Sc., Assistant Lecturer in Physics at Birkbeck College.*

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1. *Introduction.*

If a few drops of liquid be introduced between a plane mirror and a thin convex lens we have in effect a combination of the latter with a plano-concave lens of the liquid. From a knowledge of the curvature of the lower face of the convex lens together with its focal length and that of the combination, the refractive index of the liquid used can be calculated.

This method is described by T. H. Blakesley ("Proc." Opt. Con., 1905, p. 189). It is also given in "Nature," Vol. LXXXVII., Oct. 26, 1911, and in Dr. Clay's "Treatise on Practical Light," and has for long afforded a laboratory exercise, the focal distances being almost invariably measured by the pin and parallax method probably on account of the fact that the lens and mirror most conveniently lie horizontally. Thus, as usually practised the method is very rough, and no notice of it is taken by F. E. Wright in his discussion on the accuracy of the many drop methods in use (Washington Acad. Sci. "Journal" IV., pp. 269-279, June, 1914).

The object of the present communication is to show that by the choice of lenses of special curvature, and with the aid of totally reflecting prisms the method can be adapted to the optical bench and monochromatic light to give results reliable at least to the third place of decimals—an accuracy nearly always sufficient for the petrographer or the physical chemist in his determinations of molecular refractive powers. For substances having refractive index lying between 1·33 and 1·4 —which will include most aqueous solutions of salts and mixtures of alcohol and water—an accuracy in the value of μ is obtainable of about 0·0002. Thus, this method usually so roughly practised can, by the careful design of the lenses and by taking into account their thickness, be transformed with little loss of simplicity into one of the most accurate and rapid laboratory experiments on refractive index. Strictly, the apparatus here described is one for comparing refractive

indices, for it is necessary to find the constant of the instrument with a liquid of known refractive index. There is, however, no difficulty here, as the refractive index of distilled water is known at ordinary temperatures with an accuracy of practically the fifth decimal place.

2. *Simple Theory of the Method Neglecting Thickness of the Lens.*

The simple theory is well known, but is reproduced very briefly here for convenience in consequence of what follows in section 3 of this Paper.

Let f=focal length of the thin convex lens,

φ=focal length of the plano-concave lens formed of the liquid of refractive index μ_x,

F_x=focal length of the combination,

r=radius of curvature of the lower face of the convex lens, which is also that of the concave lens.

Then, by the law of combination of lenses we have numerically :—

$$1/\varphi = 1/f - 1/F_x = (\mu_x - 1)/r,$$

or,
$$\mu_x - 1 = \frac{r(F_x - f)}{F_x \cdot f}.$$

Similarly, for water or other standard substance,

$$\mu_s - 1 = \frac{r(F_s - f)}{F_s \cdot f},$$

&c.

whence
$$\frac{\mu_x - 1}{\mu_s - 1} = \frac{(F_x - f)F_s}{(F_s - f)F_x}.$$

Let $F_x - f$, which is measurable directly, $= X$.

Let $F_s - f$, likewise measurable directly, $= S$.

Then,
$$\mu_x = 1 + (\mu_s - 1)\frac{(S+f)X}{(X+f)S}.$$

On inspecting this formula it will at once be seen that the accuracy of the method depends on the measurement of differences of focal lengths. A small error made in determining f the focal length of the convex lens—which can be found once and for all—is relatively unimportant unless S and X are small or differ greatly. By using a standard liquid,

such as water, we do away with the necessity of measuring a radius of curvature and virtually find the constant K in the formula which the method assumes, viz :—

$$x=\mu_x-1=K\left(\frac{F_x-f}{F_x}\right).$$

It is of importance to find out whether the effect of the thickness of the lens is to merely modify the constant K, which for a lens infinitely thin is equal to r/f.

3. *Effect of the Thickness of the Lens.*

The effect of the thickness of the lens can be found by considering the full refraction of the parallel rays returning from normal reflection by the plane mirror.

Applying the formula :—

$$\frac{\mu}{V}-\frac{1}{U}=\frac{\mu-1}{r},$$

which holds strictly for determining "limiting foci" at spherical surface, let μ_x equal the refractive index of the liquid

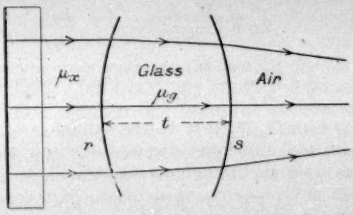

FIG. 1.

placed between the mirror and face r of the lens, and let μ_g equal the refractive index of the glass, all refractive indices being with reference to that of air taken as unity. Then, for the first refraction we have :—

$$\frac{\mu_g/\mu_x}{V'}-\frac{1}{U}=\frac{(\mu_g/\mu_x)-1}{r},$$

or, since for this refraction, $U=\infty$,

$$V'=\frac{r\cdot\mu_g}{\mu_g-\mu_x}.$$

If t be the thickness of the lens and s the radius of curvature

of the face remote from the mirror, we have for the second refraction from glass to air :—

$$U=V'+t,$$

$V=F$ (the focussing distance from the face s),

and, hence :—

$$\frac{1/\mu_g}{F}-\frac{1}{t+\frac{r}{1-\mu_x/\mu_g}}=\frac{(1/\mu_g)-1}{s}, \quad . \quad . \quad . \quad . \quad (1)$$

whereas, if air replace the liquid, x

$$\frac{1/\mu_g}{f}-\frac{1}{t+\frac{r}{1-1/\mu_g}}=\frac{(1/\mu_g)-1}{s}, \quad . \quad . \quad . \quad . \quad (1a)$$

hence,

$$\frac{F-f}{Ff}=\frac{r(\mu_x-1)}{(t+r-t/\mu_g)(t+r-t\,.\,\mu_x/\mu_g)}. \quad . \quad . \quad (2)$$

Transposing and writing x for μ_x-1, we obtain :—

$$x=K'\left(\frac{F-f}{F}\right)\left(1-\frac{t}{r+t}\cdot\frac{\mu_x}{\mu_g}\right). \quad . \quad . \quad . \quad (3)$$

where $$K'=\frac{(r+t)(r+t-t/\mu_g)}{fr}=\text{a constant.}$$

Applying formula (3) first to the substance of refractive index μ_x and, secondly, to the standard substance of refractive index μ_s, as in section 2, we obtain algebraically :—

$$x=(\mu_s-1)\left[\frac{F_x-f}{F_s-f}\cdot\frac{F}{F_x}\right]\left(1-\frac{\mu_x-\mu_s}{\frac{(r+t)\mu_g}{t}-\mu_s}\right).$$

Expressing the measurable distances of withdrawal F_x-f, F_s-f, by X and S respectively as before, and remembering that r is negative, we may write arithmetically, that is taking all symbols as positive :—

$$x=(\mu_s-1)\left[\frac{S+f}{X+f}\cdot\frac{X}{S}\right]\left(1+\frac{\mu_x-\mu_s}{T}\right), \quad . \quad . \quad . \quad . \quad (4)$$

where

$$T=\frac{(r-t)\mu_g}{t}+\mu_s. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

The use of a standard liquid thus only in part corrects for the thickness of the lens. The correcting factor, however, is small, and the denominator T is found without trouble, once and for all, in the following manner :—

The thickness of the lens t is found with a screw-gauge.

The radius of curvature of the drop face r is found without any additional measurement—with an accuracy for greater than is needed—from equation (2) above which may be written thus :—

$$\frac{x\mathrm{F}f}{\mathrm{F}-f}=\{r+t(1-1/\mu_g)\}\left\{1+\frac{t}{r}(1-\mu_x/\mu_g)\right\}, \quad . \quad . \quad . \quad . \quad (6)$$

$$=r+t(1-1/\mu_g)+t(1-\mu_x/\mu_g) \text{ very approximately.}$$

Thus, algebraically, $r=\dfrac{x\mathrm{F}f}{\mathrm{F}-f}-t(2-1/\mu_g-\mu_x/\mu_g)$ very approximately.

Whence, taking F as the focussing distance when using water, putting $x=\frac{1}{3}$ and $\mu_g=1{\cdot}5$, we have, taking all symbols with the positive sign :—

$$r=\frac{1}{3}\frac{\mathrm{F}f}{\mathrm{F}-f}+\frac{4}{9}t. \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

The accuracy obtainable in this method of measuring curvature is limited in practice by the accuracy of the measurements of f and $\mathrm{F}-f$, since in thin lenses the approximations introduce no appreciable errors. If the lens were very thick, and μ_g differed greatly from 1·5 formula (6) could always be applied, μ_g being found as described below. If $\mathrm{F}-f$ should be too small to measure accurately benzene should be used instead of water, whence, since μ_x/μ_g will be very nearly unity, we have with an even closer theoretical accuracy :—

$$r=\frac{x\mathrm{F}f}{\mathrm{F}-f}+\frac{1}{3}t. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

The only other term needed to find T is the refractive index of the glass of the lens which in a small correcting factor might be taken as 1·5. It is, however, easy to find μ_g more accurately once and for all in the following simple way. Reversing the lens, introduce between the mirror and the face of curvature s, usually uppermost, a few drops of benzene or oil of cedar wood, and find F', the focussing distance of the combination measured from the face r. The refractive index of the liquid used is not

required, but it must be near that of the glass of the lens. We have then, interchanging r and s in formula (1) :—

$$\frac{1/\mu_g}{F'}-\frac{1}{t+\dfrac{s}{1-\mu_x/\mu_g}}=\frac{1/\mu_g-1}{r}, \quad . \quad . \quad . \quad . \quad (1B)$$

or, as s is always large in suitable lenses, and μ_x/μ_g is very nearly unity :—

$$\frac{1}{F'}=\frac{1-\mu_g}{r} \text{ very approximately,}$$

whence, arithmetically, $\mu_g=1+\dfrac{r}{F'}$, ... (9)

and this value of μ_g could be used in equation (6) in finding r with precision in the case of a very thick lens.

As the refractive index of the glass of a lens is sometimes required very accurately a typical example of how this may be done is given in section 6, subsection (*d*) below :—

The approximate value of $\mu_x=1+x$, found by neglecting the correcting factor altogether, of course, serves for the numerator $\mu_x-\mu_g$ of this fraction. Typical experiments are given in section 6 below.

4. *Description of Suitable Apparatus.*

A form of suitable apparatus will be readily understood from the diagram Fig. 2. It consists essentially of two parts—an arrangement for supporting the right-angled prism P and lens and mirror M, and an optical bench and object-screen sliding piece, S. For the former purpose a good heavy adjustable "Young's Modulus table," with levelling screws, was found suitable, a small piece of metal being cut away over which the prism P stood fixed as shown. The table top T could be slightly rotated. The double platform piece D was clamped at L to the vertical table pillar, and afforded a support for the block of wood W carrying the lens and mirror. This mirror, about an inch square, could be adjusted to the horizontal by the screws s, and was cemented with red lead to the block W, which should always be replaced on the table, touching the same margin line.

The optical bench, 120 cms. long, is a good one, such as used for the bi-prism experiment, and one of its disused but well

made carrier pieces S, fitted with a vernier reading to 0·1 mm., was easily adapted for receiving an object-screen. A postcard with a small circular hole punched out of it answered this

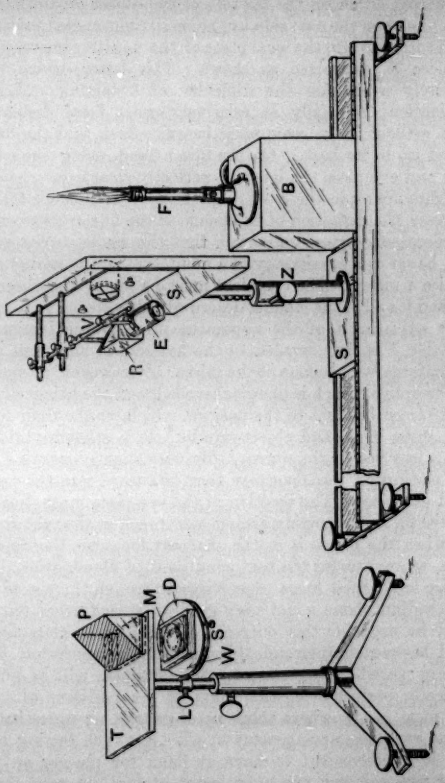

FIG. 2.

purpose well, when furnished with threads in the form of a square with diagonals, the diagonal cross-wires being very fine. This object was illuminated from behind by the sodium flame

F standing on the block B, which could easily be moved along the optical bench. When in the focal plane of the lens and drop combination, the reflected image of the cross-wires could be received either on the far side of the object on the screen itself, or else on the near side by the small right-angled prism R, which turned it into the focal plane of the negative microscopic eye-piece E, supported as shown. This latter device very effectively overcomes the difficulty of obtaining sufficient illumination, especially at relatively great focal distances, where without it the room must be quite dark, and the flame worked up to its best. On the other hand, using the small prism and eye-piece the image is perfectly clear even when all the lights are on in the room, and the sodium flame is feeble. Moreover, the definition of the image of the fine cross-wires in the microscope eye-piece is such that the whole carrier-piece S can be set to a consistency of a millimetre at a focal distance of quite a metre. The eye-piece must be set by the observer once and for all in the manner described below.

The adjustment of the apparatus prior to use is simple. The table T is first levelled to the horizontal, and then also the platform D and plane of the mirror by means of the screws *s*. The optical bench is likewise levelled, and the height of the object-screen brought to the position which, at the right focal range, gives image and object side by side in a horizontal line when a lens lies on the mirror. By very slightly rotating the table platform T the image can then be turned into the prism R and eye-piece. The position of the eye-piece must then be adjusted on its supporting arrangement, such as that shown, so that when the image is in the sharpest focus on the object-screen, as seen under the best conditions of illumination, it is likewise in sharpest focus when viewed through the eye piece. If the sodium flame is not very good an incandescent burner should be used for this purpose, a signal green glass being placed between burner and object-screen. If the prism R is properly placed the agreement, once obtained, will be found practically perfect everywhere, except for positions of very short focal range, where there is, of course, no necessity to use the eye-piece arrangement at all. For such testing purposes it is convenient to have at hand for placing on the mirror two common spectacle lenses of focal lengths about 50 cm. and 100 cm. respectively. These adjustments can be made in a few minutes, and most of them are permanent; when the eye-piece is set it should be firmly clamped at V, and

all screws tightened. Care should be taken not to move the prism P, and no means of bringing the image into its proper position on the object-screen or into the microscope eye-piece should be attempted other than a slight turn of the table T, the sliding of the entire screen piece, and if need be the slightest vertical movement of the object-screen by means of the screw Z.

In performing an experiment the lens selected is placed with the face of radius r symmetrically on the mirror, and the block W placed in position. The "zero reading" on the optical bench is then obtained when the image of the cross-wires is formed on the screen alongside the object. Two or three drops of distilled water are then placed on the mirror, and the lens replaced. A reading is now obtained through the eye-piece, and on subtracting the zero reading we obtain S, the distance of withdrawal for water. Similarly the distance of withdrawal X for the liquid under investigation is obtained. In both cases the temperature should be taken as recorded by a thermometer, subsequently placed with its bulb between the mirror and peripheral portion of the lens. Care should be taken not to warm the lens with the hand, or, at all events, time given for the temperature to recover and the focussing length to attain a steady value. Owing to the smallness of the exposed surface of the liquid evaporation is exceedingly slow; none the less it is desirable to devise a water circulation to maintain steadier conditions. The focussing length f is, of course, obtained in the ordinary manner on any optical bench employing sodium light by measuring the distance from the nearest face s of the lens to the coincident image and object. It is easily obtained correct to the nearest 0·5 mm., and is a constant, being practically independent of temperature changes.

It should be noted that the effect of the prism P is to lengthen all focussing distances by the same amount, not affecting, therefore, differences of focal lengths. Clearly actual focal lengths can be found by the periscopic arrangement by employing as a "distance-piece" any lens of known focal length and thickness.

5. *Selection of Mirror, Lenses and Standard Substances.*

I. It is absolutely essential that the small mirror, an inch square, be of good parallel plane glass—otherwise the image will show apparent astigmatism, vertical and horizontal lines

having different focal distances. There is no need, however, to resort to optically worked glass, as it is easy to select specimens of patent plane mirror glass which are quite satisfactory. Perhaps a trace of "astigmatism" is if anything helpful, enabling a sharp setting of the cross-wires for equality of definition.

II. The whole accuracy of the method is intimately connected with the careful prescription of the lenses. They should be thin, of aperture about 1 in. to 1·5 in., and made of glass of refractive index of about 1·5. In making the selection for any purpose free use should be made of the approximate formula—

$$F=fr/(r-xf),$$

and it should at the same time be remembered that while the "range of good focus" is proportional to F, the senitiveness $dF/dx=F^2/r$, and absolute measurements of short focussing distances are, therefore, to be avoided.

The following set of four lenses to which special attention has been given illustrate well the possibilities and limitations of the method. The approximate dimensions to which they were ordered are given in centimetres :—

A. A bi-convex lens of unequal radii of curvature.
$f=20$; $r=12$; $s=60$ (approx.).

B. A bi-convex lens of unequal radii of curvature.
$f=30$; $r=20$; $s=60$ (approx.).

C. A concavo-convex lens or a converging meniscus.
$f=17$; $r=7·5$; $s=64$ (approx.).

D. A bi-convex lens of unequal radii of curvature.
$f=20$; $r=15$; $s=30$ (approx.).

Experiments are given in section 6 below illustrating the respective uses of these lenses. It will be found that lenses A and B are applicable over practically the same range and thus serve well for comparative tests. The accuracy obtainable in measuring, say, a refractive index of 1·5 with lens A is dependent on the accuracy of measurement of the distance of withdrawal, X=20·96 cm. for water. An error of 1 mm. in this measurement would affect the value of $\mu-1$ for the unknown liquid by 1 part in 400. Owing, however, to the shortness of both focal distances the measurement can probably be made

correct to 0·05 cm. An error of 0·05 cm. in the measurement of f the focal length of the lens would affect the result by 1 part in 1,400. The change of focal length due to substituting benzene for water is over 50 cm.

Corresponding errors made with lens B would have less effect on the result; the correcting factor for thickness is smaller; and an error in the setting for the zero reading is of less importance. A change of focal length of some 50 cm. again results on replacing water by benzene.

Using the converging meniscus C the method is perhaps seen at its best, refractive indices up to 1·39 being certainly obtained correct to 0·0002. On substituting for water a strong salt solution of refractive index 1·3807, the focal length rose from 68 to 118 cm. Errors in measuring f or observing the zero are of minimum importance.

Lens D is specially for high refractive indices, and benzene should be used as the standard substance. For comparative purposes it is very accurate, but the absolute values are dependent on the data given for benzene below.

For ordinary laboratory use lenses C and either A or B will usually be found sufficient.

III. (*a*) Water is used as the fundamental standard substance. Its temperature coefficient is abnormally small, and its refractive index is known to the fifth decimal place. The values given below are based on those of Jamin, Lorenz, Walter and others, as given in the Physikalisch-Chemische Tabellen of Landolt, Börnstein and Roth. $\mu-1$ may thus often be taken $=1/3$ with sufficient accuracy.

Temperature.	Refractive index air to water.	Temperature coefficient.
20°C.	1·33300	0·00008
15°C.	1·33339	0·00007
10°C.	1·33369	0·00006

(*b*) Benzene will be found useful as a secondary standard, especially in determinations of high refractive indices. It can be crystallised and easily obtained pure, and its purity can be readily tested. Moreover, the presence of traces of impurity has no marked effect on the refractive index. A sample of "pure crystalline benzene" was found on a refractometer to differ in refractive index from Kahlbaum's pure benzene for molecular weights by not more than 0·0003 under constant temperature conditions. Benzene has the disad-

vantage, however, of a high temperature coefficient compared with that of water; when it is used, therefore, temperature must be carefully observed. Based on Weegmann, Knops and Landolt and Jahn we may take $\mu_{16}=1{\cdot}5039$ and $\mu_{20}=1{\cdot}5013$ with 0·00065 as the temperature coefficient. The nearness of the refractive index to that of glass is useful in determining the refractive index of the lens as explained in section 3 equation (9). For this latter purpose oil of cedar wood answers equally well.

6. *Typical Experiments on the Constants of Lenses and on Determinations of Refractive Indices.*

(*a*) Taking lenses A and B the following readings were obtained :—

TABLE I.—*Readings with Lenses A and B.*

Nature of observation.	A	B
Thickness of lens in centimetres	0·318	0·321
Focussing length correct to 0·05cm. (from nearest face *s*)	18·70	29·20
Zero on Optical Bench	4·95	15·45
Reading with oil of cedar between mirror and lens face *s*.	8·59	25·25
Readings with liquids between mirror and face *r* :—		
(*a*) Water, temperature 11·7°C.	25·91	42·52
(*b*) Chloroform, temperature 12·0°C.	51·25	69·45
(*c*) Benzene, temperatures 12·1°C., 12·2°C.	78·90	93·50

Thus, with lens A we obtain—

Distance of withdrawal S for water=20·96 ; $S+f=39{\cdot}66$.

Distance of withdrawal X for benzene=73·95 ; $X+f=92{\cdot}65$.

$\mu-1$ for water at 12°C.=0·33357,

whence regarding the lens as infinitely thin, we obtain for benzene

$$\mu-1=0{\cdot}33357\times\frac{39{\cdot}66}{20{\cdot}96}\times\frac{73{\cdot}95}{92{\cdot}65},$$

$=0{\cdot}503_{78}$, which is the very approximate value.

The denominator T of the correcting factor for thickness is now found, and will serve for always.

By formula (7) we have for the radius of curvature of the drop face *r* :—

$$r=\tfrac{1}{3}\times 18{\cdot}70\times\frac{39{\cdot}66}{20{\cdot}96}+\frac{4}{9}\times 0{\cdot}318,$$

$=11{\cdot}9_4$ cm.

And by formula (9) for the refractive index of the glass of the lens—

$$\mu_g = 1 + \frac{11{\cdot}94}{18{\cdot}70 + 8{\cdot}59 - 4{\cdot}95} = 1{\cdot}53,$$

and, hence, from (5)

$$T = \frac{11{\cdot}94 - 0{\cdot}32}{0{\cdot}318} \times 1{\cdot}53 + 1{\cdot}33,$$

$$= 57{\cdot}3.$$

Thus, we obtain for the corrected value for benzene

$$\mu - 1 = 0{\cdot}50378\left(1 + \frac{0{\cdot}5038 - 0{\cdot}3336}{57}\right),$$

$$= 0{\cdot}5038 + 0{\cdot}0015,$$

whence $\mu = 1{\cdot}505_3$ at 12·1°C.

Similarly, for chloroform,

$$\mu = 1 + 0{\cdot}4496\left(1 + \frac{0{\cdot}4496 - 0{\cdot}3336}{57}\right),$$

$$= 1{\cdot}450_5 \text{ at } 12{\cdot}0°\text{C}.$$

An Abbé refractometer with water circulation gave at 12·1°C.

for water $\mu = 1{\cdot}3334$,

for chloroform $\mu = 1{\cdot}4499$,

for benzene $\mu = 1{\cdot}5050$.

The readings for lens B give T=96·6, which could be taken as T=100, with sufficient accuracy as the correction for benzene is only 0·0009. The values obtained are :—

for chloroform $\mu = 1{\cdot}450_6$ at 12·0°C.

for benzene $\mu = 1{\cdot}505_6$ at 12·2°C.

(*b*) The converging meniscus C had focal length from near face=16·80 cm. and thickness=0·286 cm. Experiment showed $r = 7{\cdot}57$; $a\mu_g = 1{\cdot}53_5$; and T=40·4=40 approx. It is very sensitive to small changes in refractive index, and all

necessary measurements can be made with a high accuracy. A curve of the refractive index and density of salt solutions was, therefore, attempted, at the same time a comparison with an Abbé refractometer being made. The temperature of the research laboratory was very constant, and the water circulating through the Abbé prisms was easily adjusted to this temperature. A specific gravity bottle containing the solution having been weighed, drops of liquid were placed on each refractometer. The comparison is tabulated below :—

TABLE II.—*Showing Readings with Converging Meniscus C.*

Specific volume of salt sol.		Refractive index by lens and drop method.				Refractive index by Abbé refractometer.	
Weight of contents of full sp. gr. bottle.	Temperature.	Reading on optical bench.	Temperature.	$\mu-1$ uncorrected.	μ corrected.	μ	Temperature.
70·705	12·6	104·5	11·9	0·38016	1·3806	1·3809	12·0
69·416	12·8	95·92	11·8	0·37524	1·3756	1·3759	11·8
67·871	12·5	87·60	12·0	0·36968	1·3700	1·3702	12·0
66·816	13·0	82·16	11·9	0·36552	1·3658	1·3660	12·0
66·161	13·3	79·30	11·9	0·36314	1·3634	1·3636	12·0
64·549	12·6	72·37	12·1	0·35672	1·3569	1·3571	12·0
63·607	13·9	68·90	12·1	0·35310	1·3533	1·3533	12·0
62·791	12·5	65·95	11·7	0·34977	1·3499	1·3500	12·0
61·819	13·0	62·60	12·0	0·34569	1·3458	1·3459	12·0
60·955	12·9	59·81	12·6	0·34200	1·3421	1·3419	12·0
60·253	13·0	57·82	12·3	0·33919	1·3392	1·3394	12·0
59·350	13·0	55·38	12·3	0·33553	1·3355	1·3354	12·0
*58·868	12·8	54·12	12·5	...	†$1{\cdot}3335_4$	1·3334	12·0

* Water. † Assumed value.

The zero on the optical bench was at 2·95, and the reading with oil of cedar and the lens reversed was 0·30. Meniscus C can be used for concentrated common salt solutions which are at about its maximum limit. A second meniscus, E, with $f=18{\cdot}45$ and S the distance of withdrawal for water$=42{\cdot}75$ was accurate and serviceable for solutions up to $\mu=1{\cdot}41$.

(*c*) Lens D, suitable for high refractive indices gave the following data: Thickness$=0{\cdot}321$ cm.; $f=18{\cdot}80$ cm.; zero on bench$=5{\cdot}05$; reading with oil of cedar wood and lens reversed$=14{\cdot}00$; reading with water$=18{\cdot}90$ at 12·8°C.; reading with benzene$=38{\cdot}58$ at 12·6°C.; reading with carbon bi-sulphide (redistilled)$=81{\cdot}1$ at 12·0°C.

Owing to the smallness of the distance of withdrawal for water, benzene should be used as the standard, μ, being taken

as $1{\cdot}506_2$. Formula (8) gives $r=14{\cdot}96$, and T will be found $=70$ approximately. Hence, for carbon bi-sulphide :—

$$\mu-1=0{\cdot}5062\times\frac{52{\cdot}33}{33{\cdot}53}\times\frac{76{\cdot}05}{94{\cdot}85}\text{ (approx.)},$$

$$=0{\cdot}633_4,$$

$$\mu-1\text{ (corrected)}=0{\cdot}6334\left(1+\frac{0{\cdot}6334-0{\cdot}5062}{70}\right),$$

$$=0{\cdot}6334+0{\cdot}0012,$$

or $$\mu=1{\cdot}634_6.$$

An Abbé refractometer gave for benzene 1·5048 at 12·7°C., and for carbon bi-sulphide 1·6336 at 12·5°C.

(*d*) It has already been shown how by the use of a liquid of refractive index near that of glass, the refractive index of the glass itself may be easily found approximately. If the trouble be taken to find by the method of this Paper the refractive index of the liquid used, the refractive index of the glass may be found to an accuracy similar to that which holds for r and f, and hence s may be deduced.

To illustrate this the benzene found by lens B, *see* sub-section (*a*) above, to have at 12·2°C. refractive index $1{\cdot}505_6$, was found when placed between the face s and the mirror (lens reversed) to give at 12·0°C. the reading 25·15 on the optical bench. Whence $F'=29{\cdot}20+25{\cdot}15-15{\cdot}45=38{\cdot}90$.

Thus, $\mu_g=1+r/F'=1+20{\cdot}39/38{\cdot}9=1{\cdot}52_{42}$ approx.

From equation (1a) and by other means we find $s=60$ approx. But from equation (1b) we have with great accuracy for $a\mu_g$

$$\mu_g=1+\frac{r}{F'}-\frac{r}{r+s}\left(1+\frac{r}{F'}-\mu_x\right),$$

all symbols being positive. Hence,

$$\mu_g=1{\cdot}5242-\frac{20{\cdot}39}{80{\cdot}39}(1{\cdot}5242-1{\cdot}5056),$$

$$=1{\cdot}5242-0{\cdot}0047,$$

$$=1{\cdot}520.$$

From (1) or (1a) we now obtain :—

$$s=60{\cdot}4\text{ cm.}$$

(*e*) An instance of the special usefulness of the lens and drop method is the following. The author determined for

his colleague, Dr. G. Martin, the refractive indices of a number of newly-prepared alcohol derivatives of silicon hexachloride (Martin, Chem. Soc. "Trans.," Vol. CV., 1914). These compounds fuming in the air leave a deposit of white solid on glass which is only removed by warming with caustic soda. The lens can easily be warmed in hot alkali, and the ring which protects the rest of the liquid, dissolved away, but similar treatment in a refractometer would be impossible.

7. *Summary and Conclusion.*

The lens and drop method of comparing refractive indices is a very accurate one when due regard is paid to the choice of lenses, and allowance made for the thickness. It is easily adapted to the optical bench and monochromatic light, and measurements may be made very quickly. The method is perhaps seen at its best when employing a suitable converging meniscus to measure refractive indices between 1·3 and 1·4. The method of measuring short radii of curvature in terms of the well-known refractive index of water is also frequently very useful and accurate just when other means are difficult or inaccurate. The lens and drop method may also be of service in finding the refractive index of the material of the lens. The apparatus here described is very suitable as providing a laboratory experiment for advanced students in view of the accuracy of the results obtained by the inexpensive adaptation of standard apparatus.

The author must express his special thanks to Dr. R. S. Willows, M.A., of the Sir John Cass Technical Institute, for the loan of a refractometer for comparative purposes, and to Dr. A. Griffiths, head of the Physics Department, Birkbeck College, for putting at his disposal all the other apparatus needed. He would also record his indebtedness to Mr. S. D. Chalmers, M.A., for suggestions made since the communication of this Paper, especially with reference to the notation adopted.

XXXIX. *Conduction of Electricity Through Metals.* *By* Sir J. J. Thomson, *O.M., F.R.S.*

Received June 25, 1915.

The investigations of Kamerlingh Onnes on the resistance of metals at the temperature of liquid helium have led to results which are of vital importance in the theory of metallic conduction; they have shown, for example, that some metals can exist in a state where their specific resistance is less than one hundred thousand millionth part of that at 0°C. The transition from the state in which the resistance is diminishing normally with the temperature to the one where they possess this super-conductivity takes place abruptly at a definite temperature, and the difference in the electrical properties of the metal above and below this temperature are as well marked as the difference in elastic properties when a solid melts or in the magnetic ones when a piece of iron passes through the temperature of recalescence. One of the most remarkable effects discovered by Kamerlingh Onnes is that when a current, was started in a small ring of lead at a temperature of about 4 deg. absolute, by bringing a magnet close to it; the current, instead of dying away as it would have done at 0°C. as soon as the magnet was stopped, went on with practically undiminished intensity, its rate of decay being so slow that Kamerlingh Onnes estimated that it would take four days to fall to half its initial value. This power of transmitting a current for long periods when no external E.M.F. acts on the metal is one that has to be accounted for by any theory of metallic conduction; any such theory must indicate that in certain metals a change of electrical state takes place at a definite temperature, that above this temperature the current dies away almost instantaneously after the E.M.F. is removed, while below it the current may persist for days without undergoing any considerable diminution. It seems to me that this is another and fatal objection to the theory that metallic conduction is due to the presence in the metal of free electrons which drift under the electric force, for no permissible increase in the number of free electrons or in the mean free path would explain the difference between the ordinary and super-conducting state. In the case of the lead ring the maximum free path (equal to the longest chord that can be drawn in the ring) cannot be more than a few millimetres.

It is the object of this Paper to show that the effects discovered by Kamerlingh Onnes are in accordance with the theory of metallic conduction which I gave in "The Corpuscular Theory of Matter," page 86, and which, with the substitution of an electron for a charged atom is substantially the same as that given in my "Applications of Dynamics to Physics and Chemistry," 1888.

On this theory the atoms of some substances, including the metals, contain electrical doublets, *i.e.*, pairs of equal and opposite electrical charges at a small distance apart. In the normal state of a body the axes of the large number of doublets occurring in even a small volume are uniformly distributed in all directions; when, however, an electrical force acts on the body, the axes of the doublets tend to point in the direction of the force, and the moments of the doublets have a finite resultant in this direction. If the axes of the doublets were quite free to set in any direction, the smallest electrical force would be able to pull the axes of all the doublets into line and thus produce the maximum polarisation. There are, however, several influences at work which limit the number of doublets which point in the direction of the electric force.

In the case of gases, for example, there are collisions between the various molecules which tend to knock the axes of the doublets out of line as fast as they are brought into it by the electric force. Langevin has calculated from the principles of the "Kinetic Theory of Gases" the magnitude of this effect, and has shown that if M is the moment of each doublet, N the number of doublets in unit volume, I the resultant of these moments parallel to x and X the force on a doublet in this direction,

$$\mathrm{I}=\mathrm{NM}\left\{\frac{\varepsilon^{x}+\varepsilon^{-x}}{\varepsilon^{x}-\varepsilon^{-x}}-\frac{1}{x}\right\},$$

where $$x=\mathrm{MX}/\mathrm{R}\theta.$$

θ being the absolute temperature and $\mathrm{R}\theta$ the mean kinetic energy of a molecule at this temperature; when x is very small $\mathrm{I}=\frac{1}{3}\mathrm{NM}x$, when it is very large $\mathrm{I}=\mathrm{NM}$.

In the case of solids and liquids, though there may not be collisions between the molecules, the rotation of the molecules endows them with a quasi rigidity, making each molecule behave very much as if its axis of rotation were acted on by a restoring couple proportional to the angle through which the axis is displaced and proportional also to the kinetic energy possessed

by the body in virtue of its rotation, it behaves, in fact, very much like a spring whose stiffness is proportional to its kinetic energy. The value of I will be a function of the ratio of XM, the deflecting couple acting on the doublet, to the restoring couple brought into play when the axis is deflected through unit angle. As this couple is proportional to the average kinetic energy of the molecules, we have

$$I=NMF(XM/w).$$

Thus we see that for solid and liquids, as well as for gases, I is a function of MX/w.

We need not here go into the question whether the form of the function depends on whether the body is in the solid, liquid or gaseous state. It is sufficient to notice that whatever the state, when $x=0$, $F(x)=0$, and when $x=\infty$, $F(x)=1$.

Thus $F(x)$ will be represented by a curve of the type shown

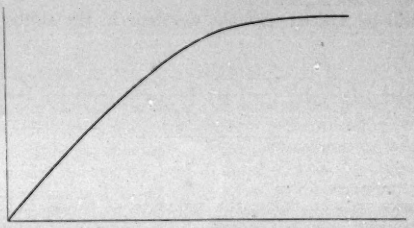

Fig. 1.

in Fig. 1. The force X which occurs in the expression for x is not merely the external electric force acting on the system, the polarised doublets will themselves give rise to strong electric forces and X is the resultant of such forces and the external electric force. We shall take the force due to the polarisation of the doublets as proportional to I and put it equal to kI. Thus if X_0 is the external electric force,

$$X=X_0+kI,$$

and

$$x=\frac{M(X_0+kI)}{w},$$

or

$$I=\frac{w}{Mk}x-\frac{X_0}{k}.$$

This relation between I and x is represented graphically by a

straight line and the value of I corresponding to any value of X_0 can be determined by finding where this line intersects the curve

$$I = NMF(x).$$

The effects corresponding to any finite value of I will be the same as if I doublets per unit volume pointed in the direction of the electric force, while the axes of the rest were uniformly distributed in all directions ; and we may picture the substance as containing a number of chains of polarised atoms whose doublets all point in the direction of the electric force as in Fig. 2.

So far as we have gone there has been nothing to differentiate between insulators and metals ; in each of these the doublets set under the electric field and give to the substance specific inductive capacity, the value of which is proportional to the value of I when X_0 is unity.

It will be noticed that the electrons in the atoms of the

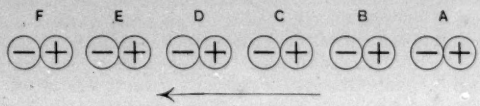

Fig. 2.

substance will be under the influence of forces excited by neighbouring polarised atoms. Thus in the case represented in the figure these forces tend to make the electrons in A move towards B and those in B to C, and so on. On this theory the peculiarity of metals is that electrons, not necessarily nor probably those in doublets, are very easily abstracted by these forces from the atoms when these are crowded together. Thus we may suppose that under these forces an electron is torn from A and goes to B, another from B going to C, and so on along the line, the electrons passing along the chain of atoms like a company in single file passing over a series of stepping stones. Let us suppose that p electrons pass along each of these chains per second, then if there are n of these chains passing through unit area at right angles to the electric force, the current i through unit area will be epn, e being the charge on an electron. If d is the distance between adjacent atoms in the chain, there will be $1/d$ atoms per unit length of chain, and I the number of doublets per unit volume pointing in the

direction of the electric force will be equal to n/d, thus $n=Id$, and therefore

$$i=epId.$$

The specific conductivity of the metal c is equal to i/X_0, so that

$$c=epdI/X_0.$$

The force exerted by the polarised atoms on the nearest electron in a neighbouring atom will be very large compared with that exerted by the external electric force, so that p will be determined by these inter-atomic forces and will not to an appreciable extent depend on the external electric force. The ratio of the current to this force will, therefore, follow the same laws as the ratio of I to the force.

We have seen that the value of I is determined by the intersection of the line,

$$I=\frac{wx}{MK}-\frac{X_0}{k}, \quad \ldots\ldots (1)$$

with the curve

$$I=NMF(x), \quad \ldots\ldots (2)$$

where w is the kinetic energy of a molecule, unless the temperature is very low $w=R\theta$, where θ is the absolute temperature and R the gas constant. When the temperature falls to the stage where the specific heat diminishes with the temperature w will be smaller than the value given by this equation.

When w/MK is considerable the line (1) will be steep and will intersect the curve near the origin, where it approximates to the straight line,

$$I=NMxF'(0). \quad \ldots\ldots (3)$$

The intersection of (1) and (3) is given by

$$I=\frac{NM^2F'(0)X_0}{w-kNM^2F'(0)},$$

and i the current by

$$i=\frac{epdNM^2F'(0)X_0}{w-kNM^2F'(0)}.$$

Thus the current is proportional to X_0 and Ohm's law holds. The specific resistance σ is given by the equation

$$\sigma=\frac{w-kNM^2F'(0)}{epdNM^2F'(0)}.$$

Now, except at very low temperatures, w is equal to $R\theta$, so that σ is expressed by an equation of the form,

$$\sigma = A(\theta - b).$$

It is thus a linear function of the temperature, which is very approximately true for pure metals.

Super Conductivity.

Let us now consider what happens when the temperature is diminished. The slope of the line (1) continually decreases and the intersection of this line with the curve gets farther and farther away from the origin. When the intersection comes on a part of the curve at an appreciable distance from the tangent

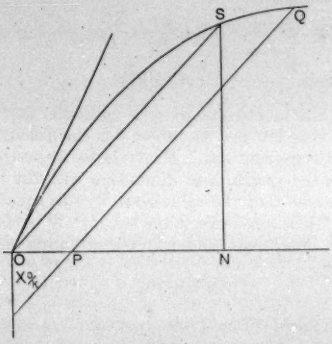

Fig. 3.

at the origin, Ohm's law will no longer hold. Suppose that the slope of the line (1) has fallen so that, as in Fig. 3, it is less than that of the tangent at the origin to the curve $I = NMF(x)$; and after the application of a force X_0, suppose the force is gradually removed, the value of I corresponding to the diminished force will be got by drawing parallels to PQ, continually getting nearer to the origin, and its value when the force has been entirely removed, by drawing a parallel through the origin itself. We see from the figure that in this case the line through

the origin will intersect the curve again at S, showing that I retains the finite value SN after the electric force has disappeared. From the point of view of this Paper, however, the part played by the electric force in metallic conduction is to polarise the metal, *i.e.*, to form chains. When once these are formed, the electricity is transmitted along them by the forces exerted by the atoms on the electrons in their neighbours. Thus, if the polarisation remains, after the electric force is removed, the current will remain, too, just as it did in Kamerlingh Onnes' experiment with the lead ring. We see that we shall have the current remaining after the removal of the electric force, *i.e.*, the metal will be in the super-conducting state as soon as the slope of the line is less than that of the tangent at the origin to the curve, *i.e.*, when

$$\frac{w}{k\mathrm{M}} \text{ is less than } \mathrm{NMF}'(0),$$

or

$$w \text{ less than } \mathrm{NM}^2k\mathrm{F}'(0).$$

Thus, the temperature at which the metal passes into the super-conducting state is such that

$$w=\mathrm{NM}^2k\mathrm{F}'(0). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

kNM is the electric force exerted by the doublets when they all point in one direction: if we denote this force by P, then w_0, the value of w at the critical temperature, is given by

$$w_0=\mathrm{MPF}'(0).$$

If the specific heat of the metal had not commenced to diminish at this temperature, θ_0, the temperature of transition into this state would be given by the equation,

$$\mathrm{R}\theta_0=\mathrm{NM}^2k\mathrm{F}'(0)=\mathrm{MPF}'(0).$$

As, however, the transition takes place at very low temperatures, when the specific heats are variable and w no longer equal to $\mathrm{R}\theta$, we must use a more general expression for w in terms of θ to determine the critical temperature. The persistence of the chains after the removal of the electric force is due to the disturbance due to thermal agitation being too weak to break up the chains when once they are formed. The chains are held together by the electric force due to the doublets in the chain itself as well as by the external electric force, and when we approach the critical temperature the force due to

the doublets is much greater than that due to the external field. We see this from the expression,

$$I=\frac{NM^2F'(0)}{w-NM^2kF'(0)}X_0.$$

This may be written as

$$I=\frac{1}{k}\frac{w_0}{w-w_0}X_0,$$

or

$$\frac{kI}{X_0}=\frac{w_0}{w-w_0}.$$

Now, kI is the part of the force on a doublet due to the other doublets, and we see from this expression that when w is nearly equal to w_0, kI is very large compared to X_0, so that the removal of X_0 will not appreciably weaken the coherence of the chains. On the other hand, at temperatures considerably above the critical, kI is small compared with X_0, so that the external force is essential for the coherence of the chains.

If the disturbing effect on the chains is entirely due to the thermal energy, and if this energy vanishes at the zero of temperature, it will always be possible to find a value of w which satisfies equation (4) and there will always be a critical temperature; *i.e.*, the metal will be able to pass into the superconducting state. It is possible, however, that the action of adjacent atoms may, independently of thermal agitation, tend to make the axes of the doublet take up a definite distribution of orientation giving a kind of crystalline arrangement to the metal, and that the doublets when disturbed from this alignment come under the action of couples tending to restore them to their original positions. We can easily take this into account. All that we have to do is to replace w in the preceding equation by $w+D$, where D is proportional to the restoring couple for unit angular displacement, due to the mutually directive action of the atoms.

The equation to the straight line (1) is now

$$I=\frac{w+D}{Mk}x-\frac{X_0}{k}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

We should expect the directive force either to be independent of the temperature or to vary but slowly with it. In this case the slope of the line will not diminish indefinitely as the temperature, but will reach a minimum value whose tangent is D/Mk. If this slope is greater than that of the tangent to the

curve at the origin, whose tangent is NMF′(0), there will be no critical temperature; hence the condition for a critical temperature is

D less than $NM^2kF'(0)$.

The restoring couple may be regarded as due to a local electrical force L exerted by the neighbouring molecules. D will then be LM, and the preceding condition may be expressed as

L less than PF′(0).

w_0' the value of w at the critical temperature is now

$$w_0' = NM^2kF'(0) - D = M(PF'(0) - L).$$

When the slope of the line is considerable, we have from equations (3) and (5)

$$I = \frac{1}{k}\frac{w_0 X_0}{w + D - w_0},$$

or

$$X_0 = \frac{k(w + D - w_0)}{w_0}\frac{i}{epd}, \quad \ldots\ldots (6)$$

and σ, the specific resistance, is equal to

$$\frac{k(w + D - w_0)}{w_0}\frac{1}{epd}. \quad \ldots\ldots (7)$$

Unless the temperature is very low, we may put $w = R\theta$, and we have

$$\sigma = k\frac{(R\theta + D - w_0)}{w_0}\frac{1}{epd},$$

if σ_0 is the resistance at 0°C. and α the temperature coefficient of the resistance,

$$\sigma_1 = \sigma_0(1 + \alpha t),$$

where t is the Centigrade temperature. Comparing this with the previous expression, we see that

$$\alpha = \frac{1}{273 + \frac{D - w_0}{R}}.$$

The condition for the existence of a critical temperature is $D < w_0$, *i.e.*, that the temperature coefficient of the resistance when the temperature is not very low should be greater than 1/273.

When D is considerable the line (5) will be steep, so that at all temperatures the intersection of the curve and the line will

be quite close to the origin. We may, therefore, use equation (3) even for very low temperatures, so that at all temperatures,

$$\sigma = \frac{k}{epd} \frac{(w + \mathrm{D} - w_0)}{w_0}.$$

The temperature coefficient of the resistance is proportional to $d\sigma/d\theta$, and this, as we see, is proportional to $dw/d\theta$. This quantity, the rate of increase of the energy with the temperature, is proportional to the specific heat at the temperature θ.

As the specific heats of many substances are very much smaller at the low temperatures obtained by the use of liquid hydrogen or helium, than at normal temperatures, we see that on this theory the temperature coefficients of metals which have no critical temperature ought to be very small at low temperatures. The experiments of Kamerlingh Onnes and Dewar and Fleming show that this is in some cases a very well marked effect. Fig. 4 shows the variation of resistance of

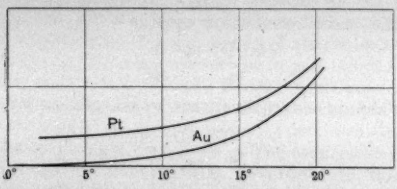

Fig. 4.

gold and platinum suspected of not being quite pure; it will be noticed that at very low temperatures the resistance becomes almost independent of the temperature. Similar effects are shown by many alloys; they would, on the theory, be shown by any metal or mixture which had not too small a value of D and whose specific heat fell appreciably at low temperatures.

In fact, the general behaviour of alloys seems to admit of a satisfactory explanation on the supposition that in them, or at any rate in those whose resistance is considerably greater than the value calculated from their percentage composition, the restoring couple D is much greater than in pure metals. This seems what we might expect when the alloy is not a mere mixture; for if it was a definite compound of the two metals

we should expect that there would be a tendency for the axes of the molecules of one metal to have definite orientation with reference to those of the molecules of the other. The same thing would also apply if the metals did not form definite compounds with each other but did form crystals.

We see from the preceding equations that if D were large for these alloys they would have (1) a small temperature coefficient at normal temperatures and a very small one indeed at temperatures low enough to diminish the specific heats, (2) they would not have a critical temperature and would never pass into the super-conducting state. These are characteristic properties of the resistance of alloys.

Again, if there are m molecules of one metal, n of the other per unit volume we should from the expressions (B) for the specific resistance of a pure metal expect that the specific resistance of the alloy would be given by a formula of the type

$$\sigma=\left\{\frac{mk_1}{ep_1d_1}\frac{(w+D_1-\omega_0)}{w_0}+\frac{nk_2}{ep_2d_2}\frac{(w+D_2-w'_0)}{w'_0}\right\}/(m+n).$$

As this involves the restoring couples D_1, D_2 it cannot be calculated from the resistances of the metals when pure; we see, however, that $\sigma_t-\sigma_T$ the difference in the specific resistances of the alloy at the temperatures t and T is given by the equation

$$\sigma_t-\sigma_T=\left\{\frac{mk_1}{ep_1d_1}\frac{(w_t-w_T)}{w_0}+\frac{nk_2}{ep_2d_2}\frac{(\omega-T)}{w'_0}\right\}/(m+n),$$

the D's have disappeared from this equation, and it is exactly the value we should have calculated from the resistance of the metals separately. This is the result known as Matthiessen's rule, which states that even when the specific resistance of the alloy can not, the difference between the specific resistance at two temperatures can be calculated from its constituents. So far we have supposed D is independent of the temperature; if it changes appreciably with it, as it might be expected to do if the nature of the compounds, or mixed crystals formed by the two metals, did so, the temperature coefficients would show anomalies such as those found in alloys which have negative temperature coefficients.

I have shown ("Corpuscular Theory of Matter," p. 86) that the electric and thermal conductivities will on this theory bear a nearly constant ratio to each other if the electrons which take part in the conduction are in thermal equilibrium with the metal in their neighbourhood.

ABSTRACT.

The discovery by Kamerlingh Onnes, that at the temperature of liquid helium some metals can exist in a state in which their specific resistance is less than one hundred thousand millionth part of that at 0°C., appears to necessitate the abandonment of the ordinary theory of metallic conduction, as the experimental conditions prohibit the explanation of the phenomenon by an abnormal increase, either in the number or mean free path of the free electrons. The effects observed by Kamerlingh Onnes may, however, be accounted for by a theory of metallic conduction previously given by the author in "The Corpuscular Theory of Matter." On this theory the atoms of some substances contain electrical doublets—*i.e.*, pairs of equal and opposite electrical charges at a small distance apart. The effect of an applied E.M.F. is to alter the heterogeneous distribution of the axes of these doublets by bringing them into partial alignment with the field. The influences preventing complete alignment are considered, and it is shown that if M is the moment of a doublet, N the number per unit volume, w the average kinetic energy of the molecules ($=\mathrm{F}\theta$, except at very low temperatures) and I is the resultant of the molecular moments in the direction of X the electric force on the doublets, then

$$\mathrm{I}=\mathrm{NMF}(\mathrm{XM}/w)=\mathrm{NMF}(x), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

in which $\mathrm{F}(x)=0$ when $x=0$, and $\mathrm{F}(x)=\infty$ when x is infinite.

X is made up of the applied electric field X_0 plus an internal field due to the polarised doublets, the latter of which is assumed to be proportional to I. Hence, $\mathrm{X}=\mathrm{X}_0+k\mathrm{I}$, and $x=\mathrm{M}(\mathrm{X}_0+k\mathrm{I})/w$, or

$$\mathrm{I}=wx/\mathrm{M}k-\mathrm{X}_0/k. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

For any value of X_0 the value of I can be found from the intercept of the straight line (2) with the curve (1). The effects due to any value of I will be the same as if I doublets per unit volume pointed in the direction of the field, the axes of the rest being uniformly distributed in all directions, and the substance may be pictured as containing a number of chains of polarised atoms whose doublets all point in the direction of the field. The electrons in the atoms will be acted on by forces due to the neighbouring polarised atoms, and the theory supposes that in conductors the electrons are easily abstracted by these forces from the atoms to which they are attached, and pass, under their influence, from atom to atom round the polarised chain. If p electrons pass along each chain per second, and if there are n chains per square centimetre perpendicular to the field, the current density $i=epn$, where e is the electronic charge. It is shown that p is independent of X_0, and so the ratio of i to X_0 will follow the same laws as that of I to X_0.

When $w/\mathrm{M}k$ is large, as at ordinary temperatures, the slope of (2) will be steep and will intersect (1) near the origin where it approximates to a straight line. In these circumstances it is shown that Ohm's law holds. As the temperature falls the slope of (2) also decreases, and may ultimately become less than that of the tangent at the origin of (1). In this case, if we start with an external field producing a polarisation, I, and gradually reduce the field to

zero, the point of intersection of (1) and (2) moves along the former, but still gives a finite value of I, when (1) passes through the origin—*i.e.*, when $X_0=0$—and a current continues to flow in the absence of an applied E.M.F. as observed in one of Kamerlingh Onnes' experiments.

On this view, therefore, the function of the applied field is to produce the alignment of the doublets; the actual transference of electricity is effected by the large inter-atomic forces brought into being by the polarisation of the doublets. Thus, if the polarisation remains on withdrawing the applied E.M.F. the current will also remain.

In addition to the disturbing effects on the chains due to thermal energy, there may be mutually directive action between different atoms such as gives rise to crystallisation. The effect of this is considered, and it is shown that if this factor is large the metal cannot become superconducting.

DISCUSSION.

Prof. S. P. Thompson, in proposing a vote of thanks to the President for his Paper, said there were one or two points which he had not quite grasped in the course of the lecture, and which he would like to have cleared up. We were asked to assume a ring of polarised atoms with positive and negative sides producing an electric force which caused the transference of electrons from one atom to the next and so on round the circuit, this transference constituting the electric current. Although the responsible factor in the transference of the electrons was the internal force, an external force was necessary to produce this, and he did not see why, on withdrawing the applied field the process should go on instead of stopping after one, or, at most a few, transfers had taken place. Were there any physical grounds for supposing that the forces of restitution were greater in the case of mixed crystals than in homogeneous ones? Lastly, he did not see the physical necessity of introducing the quantity D. Could the total force of restitution from all causes not have been included in a single symbol?

The President, replying to the points raised by Prof. Thompson, said that the energy was not spent in the movements of the electrons, but in creating the initial polarisation, and if, in any circumstances, this could be maintained, the current would go on without any loss of energy, except a little by radiation. The formation of mixed crystals of A and B was due to the force between a molecule of A and one of B being greater than that between two of A or two of B. Hence, the force of restitution called into play on displacing a molecule will be greater in the case of the mixture than in that of the pure metals. He had thought it better to denote the force of restitution due to the directive action of neighbouring molecules, which did not depend much on temperature, by a separate symbol D, and keep it clear from the restoring couple due to the gyrostatic action which did depend on the thermal conditions.

XL. *On an Unbroken Alternating Current for Cable Telegraphy.* *By* Lieut.-Colonel George O. Squier, *Ph.D.*

Received May 28, 1915.

I. *Introduction.*

The object of this Paper is to propose a new angle of view in the method of transmission of signals in the submarine telegraph cable, and to describe some apparatus for operating on the general principles involved.

Although more particularly an engineering subject, yet it is brought before the Physical Society in the hope that some of its members may see in the plan proposed some points in the fundamental theory of ocean cabling for further research.

The phenomenal progress of wireless telegraphy has been made possible only by the combined efforts of some of the world's best-equipped physicists and the practical engineer. It is in the hope of a similar co-operation of effort for the advancement of cable engineering practice that I invite your attention at this early stage of development.

Experiments have been conducted during the past two years at the works of Messrs. Muirhead & Co. (Ltd.) with a view of determining the practical application of the sine-wave type of E.M.F. for cable signalling. The results thus far obtained have only been made possible through the wide range of experience and practice which this distinguished firm of cable engineers have been able to bring to bear on the subject, and they have taken a leading part throughout the experiments carried out.

Sixteen years ago Dr. Crehore and the author conducted some experiments on an Atlantic cable from Waterville, Ireland, to Nova Scotia, Canada, using a special form of dynamo as a source of power for operating the cable.

A transmitter was also devised, in which a special feature was the cutting out or suppressing from the alternating current certain definite semi-waves of current which enabled the cable code to be transmitted.

At that time the demand for increase of speed over ocean cables was not pressing. In fact, the particular cable used was idle for several hours each day, which fact, however, greatly facilitated the opportunities for experimenting.

Wireless telegraphy did not then exist, and the cable relay or amplifier had not appeared, so that very accurate balancing of the duplex-bridge was not required.

Conditions have changed materially since 1899, until at the present moment, it may be said that the matter of obtaining increased speed on ocean cables with present apparatus, is largely controlled by the accuracy with which the duplex-bridge may be balanced.

II. *An Ocean Cable Considered as a Power Line.*

If an engineer were required to design a system for operating an electric motor through an Atlantic cable, no form of generator could be proposed at present, other than a single-phase alternating current of the sine-wave type. This form, we know, will deliver power at the receiving end of the cable more efficiently than any other shape of wave. Furthermore, during the operation of such a motor, the generator would be allowed to run smoothly and regularly, and, in particular, the generator circuit would never be metallically opened or closed during operation.

The opening and closing of an alternating current circuit is well known to produce disturbances of a more or less pronounced character, depending upon the angle of phase at which the current is opened or closed.

Since improvement of the duplex-bridge balance, as stated above, is really at present a most important desideratum, experiments were made using the Muirhead artificial cable to compare the present forms of battery transmitters with various modifications of the sine wave of E.M.F.

In the last analysis it was always the opening and closing of the transmitter circuit which produced the final kick or "jar" in the balance. This is not surprising when we remember that, from an electrical standpoint, few things can be done to a circuit more severe than suddenly to introduce an infinite resistance into it.

However, since there is an appreciable spark or arc at the instant of opening the battery transmitter circuit, this discharge controls to a greater or less degree the character of the break; but the exact influence of this is not so easy to predict, for, though it is probable that the current dies away quicker with a sudden break than it does with a very slow one, which permits the arc to remain for some time, yet it is not proved certainly, for the more rapidly the break is made the faster

the resistance increases, and therefore, probably, the rate of change of current, and with it the counter E.M.F., increases. An increased E.M.F. can bridge a longer gap, but a longer gap may be made in the same time that a short one is, with a less velocity at the break. So it appears that these two considerations counteract each other, and it all depends upon which has the greater influence.

It was only after many variations of the simple alternating current had been tried in the course of experiments to perfect the duplex-balance of the bridge that the fundamental principle of never breaking the transmitter circuit became impressed.

Continuing the analogy of the power plant, it may be remarked that practically every form of cable recorder, amplifier or relay is essentially an alternating-current motor. Its field

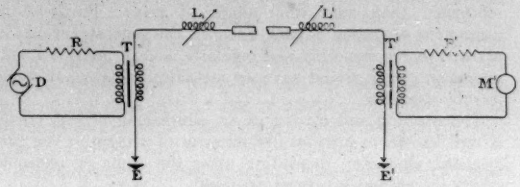

FIG. 1.—SIMPLEX CIRCUIT FOR OCEAN CABLE CONSIDERED AS A POWER LINE.

magnets, armature coil and the counter E.M.F. of damping are subject to the same laws as in the motor for power purposes.

Let us assume, therefore, as a starting point, the standard type of circuit for operating a cable simply as a power plant as the ideal solution of the problem, and then determine by experiment how near this solution may be retained in practice for the purpose of transmitting cable signals according to the present alphabet. In this plan it will be the continued experimental purpose to determine the minimum possible variations in such a system to enable the alternating current received to be interpreted into dots, dashes and spaces.

A siphon recorder placed in the receiving end of such a cable plant would trace on the recording slip an uninterrupted sine-wave of current which may be considered as the theoretically perfect form of siphon record always to be aimed at, although never to be actually attained in practical telegraphy.

In Fig. 1, D represents a single-phase alternating-current

dynamo, R and R′ are resistances and M′ is a motor to be operated.

Since a circuit comprising an ocean cable is largely overloaded with capacity, and already includes a large ohmic resistance, it will be in the direction of increasing the resultant harmonic current flowing in such a circuit, to insert in the sending end of the cable a variable inductance, L_1. For symmetry, L′ is a variable inductance similar to L_1.

T and T′ are iron-cored transformers or auto-transformers, the coils of which are of low resistance, and connected directly to earth at E and E′.

The circuits in Fig. 1 are closed circuits throughout.

For duplex working Fig. 1 becomes Fig. 2, in which L_1, L_2,

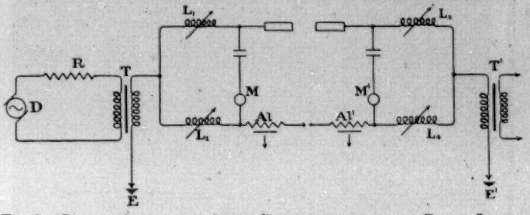

FIG. 2.—DUPLEX CIRCUIT FOR OCEAN CABLE CONSIDERED AS A POWER LINE.

L_3, L_4 are the inductance arms of the bridge, and the motors M and M′ are inserted in the usual manner for cable working; *Al* and *Al′* are the artificial lines.

It now remains to inquire as to what modifications must be made in the typical duplex power-circuit in Fig. 2 to enable the motor recorders to indicate the elements of the cable alphabet. These elements are three in number, and only three—*i.e.*, the dot, the dash and the space—and each of them is equally important in interpreting the record, and for this reason they will each be considered in the general sense as signal units, rather than the usual way of regarding only the dots and dashes as the signals, and not the spaces.

It is also usual to speak of cable speeds in terms of standard letters per minute transmitted; but for our present purposes it will be more convenient to convert this speed into terms of the frequency of the dynamo. The graph (Fig. 3) exhibits the linear relation between n, the frequency of the dynamo, and letters per minute transmitted, based on the assumption that

the average cable letter, with its space, requires four units of alphabet time.

Careful experiments on an Atlantic cable confirm the theory of the subject—that, no matter what the shape of the alternating current transmitted, approximate sine waves are received at the distant end of the cable. Battery reversals

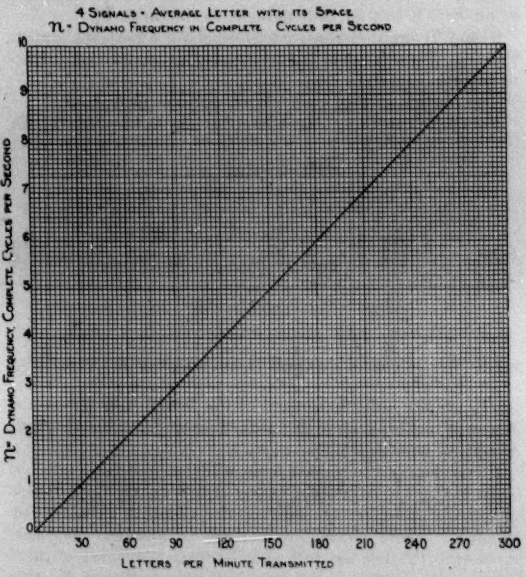

Fig. 3.—Speed of Cable Signalling in Letters per Minute in Terms of the Frequency of the Alternator.

produce just as accurate sine-wave signals on the receiving recorder as does the alternator itself.

In other words, it is the fundamental term of the Fourier analysis which is alone concerned in making the record at the receiving end of the cable.

Since, therefore, both theory and experiment show that a sine form of wave is the only one which can pass through the

cable without changing its characteristic shape, it should be an advantage so to alter the typical power circuit in Fig. 2 as to preserve the sine characteristic as far as possible.

This can be done in a simple manner by operating upon the primary circuit of the transmitter containing the generator, to alter the impedance of the circuit in synchronism with the generator itself. The problem is the more simple from the practical standpoint, because we are dealing with frequencies from about 4 to 10 cycles per second, and, therefore, it is easy to operate with great accuracy on this primary current at any angle of phase. If the change of impedance of the primary circuit always takes place at the instants when the current flowing in the circuit is naturally zero, the fundamental frequency of the current will not be changed, and the sine characteristic of the wave will be very approximately maintained.

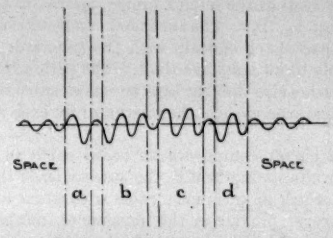

FIG. 4.—CURRENT WAVES FOR TRANSMITTING THE LETTERS *a*, *b*, *c* AND *d* BY THE ALTERNATING CURRENT.

Varying the impedance of the primary circuit will change the amplitude of the individual alternations of the current.

Another way of stating the effect is that the voltage between the end of the cable and the earth is made to vary by this process, to indicate the three elements of the cable alphabet. It should be noted that this plan transmits alternating current for the spaces between letters and words with exactly the same regularity and integrity as for the signals themselves.

Dots, dashes and spaces a.e each transmitted by impulses of either sign, differing from each other only in amplitude.

Fig. 4 shows diagrammatically the form of current waves transmitted for the letters *a*, *b*, *c* and *d* with the accompanying spaces required.

Although an alternating current operated upon as above

outlined could be read directly from a standard siphon record, yet it would be considered an advantage if the final received record could be printed in Morse characters on the usual receiver slip. To accomplish this it will be necessary to provide an apparatus which operates solely by the amplitude of the current waves received, and not by their sign. Furthermore, since the impulses transmitted and received for the spaces are not required on the printed slip, the apparatus should be capable of sorting out and omitting these parts of the record automatically.

Description of Circuits Used.

Fig. 5 shows diagrammatically one form of circuit arrangement based on the above principles for transmitting and receiving messages. The usual transmitting condensers are shown in the arms of the duplex bridge, instead of the inductances L_1, L_2, L_3, L_4. The ordinary transmitting tape is caused to move synchronously with the generator by being geared directly to its armature shaft. The perforations in the tape are of such a size that the tape advances a distance corresponding to one semi-cycle of the alternator for each individual perforation.

Since in a circuit comprising an ocean cable on which is impressed an alternating E.M.F. the current leads the E.M.F. by an angle which in an infinite cable is constant at 45 deg., it is necessary to provide in the transmitter, mechanism for moving the tape carriage longitudinally relative to the perforations in the tape. A micrometer screw is provided for this purpose, and by its use the current can be operated upon accurately at any angle of its phase, and in practice it is so adjusted that the impedance of the primary circuit will be changed at the instants of zero phase of the current. Since also the angle by which the current leads the E.M.F. is dependent upon the resistance of the primary circuit, there is also provided in addition an adjustment of the transmitter for the very slight difference of phase for the dot-and-dash conditions, so that in effect all operations on the primary current, whether for a dot, dash or space, are adjusted to take place at the zero point of current.

The general operation of the transmitter mechanism is as follows : When no holes are perforated in the tape t, the whole resistance R is in the primary circuit and an alternating current is transmitted, which corresponds to spaces between

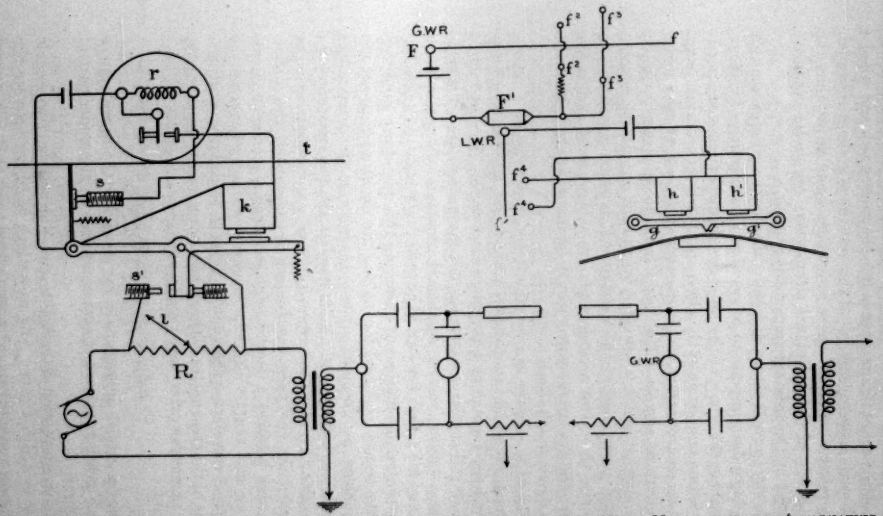

FIG. 5.—ONE FORM OF DUPLEX CIRCUIT ARRANGEMENT FOR TRANSMITTING AND RECEIVING MESSAGES BY THE ALTERNATING CURRENT.

letters and words, and when a perforation in the tape is on the dot side, part of this resistance R is short-circuited, and when a dash perforation occurs, all of this resistance is short-circuited. The relative values of these resistances being adjustable, the amplitude of the individual alternations are under complete control by the ordinary transmitting tape.

In the figure is shown diagrammatically the arrangement for causing a dash impulse to be sent by the transmitter by short-circuiting the whole of the resistance R for a time corresponding to one semi-cycle of the alternator. r is an ordinary relay, k is an electromagnet which operates at the end of each signal or group of signals. The line l leads to an exactly similar arrangement which is provided for the dot impulse by which a certain proportion of the resistance R is short-circuited. In practice these two arrangements are side by side, and in the figure the dot arrangement would be immediately behind the one shown for the dash.

The tape carriage can be moved longitudinally by a micrometer screw which is not shown in the figure, and has an adjustment over a range of a complete semi-cycle. The contacts s and s', and similar ones for the dot mechanism are adjustable by slow-motion screws, so that the instants of contact for a dot and for a dash have an adjustment relatively to each other

A convenient method of observing the wave-form produced by the transmitter, and for making the above adjustments, is to include an ordinary siphon recorder in the transmitter circuit, or, better, in the cable itself at the transmitting end; by connecting it at the terminals of a very low resistance in the circuit. At the very low frequencies involved, the siphon recorder becomes a most useful and accurate current curve-tracer, and enables the experimenter to observe exactly the shape of wave being transmitted into the cable for any adjustment of the transmitter. Once these adjustments are made for any particular cable they remain unchanged.

Form of Receiving Circuits.

One practical arrangement for receiving the signals on a Morse printer is shown diagrammatically in the upper part of the figure.

F is an adaptation of the well-known Muirhead gold-wire cable relay, in which f is the gold wire which oscillates between the platinum contact posts $f^3 f^3$, about a fixed point at F. A second pair of posts $f^2 f^2$ is provided, and each of these pairs has an

adjustment for altering their distance apart and also for moving them longitudinally along the gold wire and relatively to each other. Instead of being separate contacts, as in the usual case with this relay, each of the pairs are electrically connected as shown. F′ is a local wire relay of similar principle, and f' is its moving part which in its oscillations makes contact with the posts $f^4/^4$ adjustable as above; h and h' are electromagnets for operating the dot and dash arms of the printer; g and g' are two arms pivoted at their outer ends, adapted to make a siphon record at their free ends in the centre upon the moving tape shown beneath. The arm g' is forked to record two ink marks on the tape symmetrically on either side of the central mark made by the arm g, and in line with it across the tape, so that both arms can make a record simultaneously on the tape for a dash.

The incoming alternating current thus causes the gold wire f to oscillate back and forth with different amplitudes depending upon whether dots, dashes or spaces are being received, and the posts $f^2/^2$ are adjusted so that the space amplitude just does *not* make contact, but does make contact for a dot amplitude of either sign. In like manner the posts $f^3/^3$ are adjusted so as just *not* to record a dot impulse, but to make contact for the larger amplitude of a dash of either sign. It is seen that both a dot and a dash contact are made whenever a dash amplitude is received. The flexible gold wire f in its motions about F first strikes the posts $f^2/^2$, and there is a bending or wrapping effect produced in the wire, which for the stronger dash impulses causes contacts with $f^3/^3$ also. The relay F′ operates similarly through a moving arm, making contact with the posts $f^4/^4$ separately or both together, depending upon the amplitude of its swing.

The printer itself becomes an apparatus of marked simplicity, and is nothing more than two small siphons adapted to mark on the slip in the usual manner.

In practice, all adjustments are so made as to provide for transmitting for the spaces as large an amplitude as possible instead of as small an amplitude as possible, in order to approximate more nearly to the ideal electrical conditions for transmission through the cable itself.

In case a cable magnifier such as the Heurtley instrument is used instead of a gold-wire relay, it would be inserted in Fig. 5 in the recorder arm of the bridge, and operate a local wire relay and printer instead of the usual recorder. Since the

relay operates on difference of amplitude of the waves, and not on their absolute value, a cable magnifier would serve to exalt the value of this difference and furnish a greater margin for practical working.

Resolving Power or Definition.

The ideal alphabet to employ in cable signalling would be one in which other things being equal, each letter had the same limit of legibility; because if this is not so, the speed of signalling is lowered to meet the legibility of certain letters only. In present cable practice, what may be called the "resolving power," or the definition, is not equal throughout the letters of the alphabet.

Letters like "a" or "n," for instance, may be considered as perfect letters, and whether sent by one complete cycle of E.M.F. of a dynamo, or by two square-topped waves of equal area and opposite sign separated by a time interval, produce on the siphon record approximate sine-waves. On the other hand, letters like "s" or "h," where three or four square-topped waves of the same sign are sent into the cable consecutively, the received record at high speed fails to resolve these separate impulses, and the siphon record becomes a more or less continuous hump of large amplitude, which the expert operator learns to read without being able to detect the individual impulses; in fact, some practical operators seem to prefer these letters to what are known as "cross-letters," such as "a" and "n."

Careful experiments have been conducted through long cables, which prove, however, that the so-called "cross-letters" have a superior legibility, which obviously should be the case from theoretical considerations only.

We may consider the siphon record of an uninterrupted alternating current as possessing 100 per cent definition, independent of the frequency, the voltage employed, or the particular cable used, and regard it as the standard of definition for all signals. Indeed, such a record is, in fact, a message composed of a series of the letter "a" joined together without spaces between.

In present practice it is found necessary to insert a receiving condenser in series with the recorder coil, or an inductive shunt around the coil, for the purpose of improving the definition of certain letters of the alphabet, such as "s" and "h," as well as to eliminate from the record the effects of earth currents of very low frequency induced in the cable.

III. *Electrical Measurements of the Cable Circuits.*

One of the chief attractions in the use of an unbroken alternating current for operating the cable is the fact that for the first time we are enabled to measure the constants of the different elements of the bridge and cable circuits employed under actual signalling conditions. Since one particular frequency is used and the current is never broken, it is necessary and sufficient to use the ordinary commercial hot-wire ammeters and voltmeters which are now available over suitable ranges.

A tachometer is attached to the dynamo armature shaft, so that the frequency is indicated continuously, and these readings, in connection with those of the ammeter and voltmeter, are all that are required to determine the impedance and phase angle of any part of the transmitter circuit, including those of the cable itself.

In the present forms of battery transmitter a square-topped wave is employed, and, in addition, it is found necessary to disconnect the battery and connect the cable to earth during a portion of each individual signal sent. The reason for this is that, in this form of wave, the cable receives a charge dependent upon the time during which the key remains closed for the signal, and when the circuit is opened it is necessary to give time for the cable to become discharged before the succeeding signal can be sent. This discharge is shown in the spark that is seen in the present form of transmitter when the circuit is broken. The percentage of time of each elementary signal during which the cable is connected to earth varies according to the cable, but in long cables it is usually about 25 per cent. of the whole time of the signal.

This means that of necessity the cable is entirely disconnected from the battery at both ends for a period aggregating six hours per day, due to the present method of sending the individual signals.

This discharge of the cable has, indeed, been turned to account for a useful purpose by the late Mr. Gott,* who quite recently developed a system of transmission in which the discharge of the cable is ingeniously utilised to operate the tongue of a relay, which causes the succeeding signal to enter the cable with the opposite sign.

This form of square-topped wave, when combined to make up letters and words, causes a broken and irregular form of

* British patents No. 10,534 and No. 22,364 of 1912.

current to enter the cable, which precludes the possibility of measuring the ordinary quantities, such as voltage, current, impedance and phase-angle of the different elements of the circuit. This is extremely unsatisfactory from an engineering standpoint, and retards progress.

The development of the modern artificial line for simulating the action of the cable has required a large amount of patient and careful work extending over a number of years. As soon as any form of break, with its consequent spark or arc, is removed from the transmitter circuit, and an alternating current of one frequency only is substituted, the artificial line can more faithfully represent the action of the current flowing in the real cable, and its construction may be simplified.

Transmitting Impedance of an Atlantic Cable.

In the case of long submarine cables having resistance and distributed capacity, self-induction and leakance being neglected, we have the well-known sine-wave formulæ for an infinite cable

$$e=\mathrm{E}\varepsilon^{-\sqrt{\frac{\mathrm{CR}\omega}{2}}x}\sin\left\{\omega t-\sqrt{\frac{\mathrm{CR}\omega}{2}}x\right\}, \quad . \quad . \quad . \quad . \quad . \quad (1)$$

$$i=\mathrm{E}\sqrt{\frac{\mathrm{C}\omega}{\mathrm{R}}}\,\varepsilon^{-\sqrt{\frac{\mathrm{CR}\omega}{2}}x}\sin\left\{\omega t-\sqrt{\frac{\mathrm{CR}\omega}{2}}x+\frac{\pi}{4}\right\}, \quad . \quad (2)$$

e and i represent instantaneous values of the voltage and current at any point of the cable at a distance x from the origin and a time t: E is the maximum value of the E.M.F. applied to the cable, R and C are the resistance and capacity of the cable per unit length, ω is 2π times the frequency.

These formulæ are accurate for all practical purposes in the case of long cables, as will be shown presently.

At the transmitting end of the cable $x=0$, and (2) becomes

$$i=\mathrm{E}\sqrt{\frac{\mathrm{C}\omega}{\mathrm{R}}}\sin\{\omega t+45^\circ\}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

$$i=\mathrm{I}\sin[\omega t+\theta], \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

where I is the maximum current and θ is the angle by which the current leads the E.M.F. This angle is constant and equal to 45 deg. The transmitting impedance of the cable is

$$\mathrm{Z}_1=\frac{\mathrm{E}}{\mathrm{I}}=\sqrt{\frac{\mathrm{R}}{\mathrm{C}\omega}}=\sqrt{\frac{\mathrm{R}}{2\pi\mathrm{C}}}\sqrt{\frac{1}{n}}=c_1\sqrt{\frac{1}{n}}, \quad . \quad . \quad (5)$$

where c_1 is constant for any particular cable.

$$I=\frac{E}{Z_1}=\frac{E}{c_1}\sqrt{n}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

Let us assume

$$R=4{,}895 \text{ ohms.}$$
$$C=914\cdot10^{-6} \text{ farads.}$$
$$\text{Length}=2{,}164 \text{ nautical miles,}$$

data which represent one of the best transatlantic cables.

Substituting in (5) and (6) we have

$$Z_1=\frac{923\cdot2}{\sqrt{n}}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

$$I=\frac{E}{923\cdot2}\sqrt{n}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

The graph (Fig. 6) represents the impedance-frequency curve

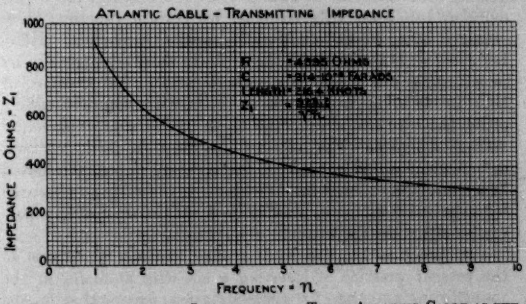

FIG. 6.—TRANSMITTING IMPEDANCE OF A TRANS-ATLANTIC CABLE AS THE SPEED OF SIGNALLING IS VARIED.

at the transmitting end of this particular cable plotted from (7).

For $n=5$, or at 150 letters per minute, the cable impedance is 412 ohms, or only 8·4 per cent. of the ohmic resistance of the cable.

The impedance of this same cable, as measured by the ammeter-voltmeter method, for a frequency of $n=5{\cdot}03$ is 397·5 ohms.

With 50 volts on this cable at the same speed of transmission the current flowing into the cable from (8) is

$$I=0{\cdot}12 \text{ ampere.}$$

The transmitting impedance is most sensitive to change of frequency at the lower frequencies, and by increasing the speed from $n=4$, or 120 letters per minute, to $n=9$, or 270 letters per minute, the cable impedance is decreased in the ratio of 3 to 2, and the current correspondingly increased. The same values of impedance and current deduced above apply to the equivalent artificial line, which is made to balance as far as possible the real cable.

By inserting a hot-wire ammeter in the cable itself beyond the bridge, and also connecting the transmitting end of the cable to earth, through a suitable voltmeter (preferably an electrostatic instrument, to prevent any disturbance of the bridge balance), we have the means at hand for determining the best transmitting conditions for any particular cable and frequency of signalling. Heretofore the value of the transmitting condensers in the duplex arms of the bridge have been more or less arbitrarily assigned by certain practical rules obtained by an expert study of the signals themselves. Here, however, we may approach the problem more scientifically by so adjusting the values of the condensers or inductances, or both, in the bridge branches as to produce a maximum reading of the hot-wire ammeter in the cable itself for any particular voltage assigned. The criterion for best transmitting conditions alone would be that the transmitting impedance should be a minimum, or the current flowing into the cable as measured by a hot-wire ammeter should be a maximum for any particular signalling frequency.

This practical method is, of course, entirely analagous to the present practice of inserting a hot-wire ammeter in the transmitting antenna of a wireless station, except that in the latter case we can go much further and adjust for the maximum current possible at the maximum point of the resonance curve of the antenna. In the cable circuit we are dealing, not with resonance, but with forced waves maintained by the dynamo.

Electrical Stress Upon the Cable.

Submarine cables are at present operated by primary batteries giving an open-circuit voltage of from 50 to 80. The main reason for this is, of course, the constant fear of sub-

jecting the cable itself to undue electrical strain, particularly in the deep sea portion. where repair is difficult and expensive. This is not the only reason, however, although it is the principal one.

The present form of battery transmitter impresses upon the cable square-topped waves, which may be analysed by Fourier's method, and in the hands of Malcolm and a few others the wave-form has been worked out and predicted as the wave passed through the cable to the receiving end. Since we know that it is only the fundamental term of the Fourier analysis which produces an appreciable effect at the receiving end, all of the harmonic waves which are required to build up the square-topped form of wave are impressed upon the cable at the transmitting end, and are absorbed in the cable itself, and never reach the receiving end. These, therefore, represent superfluous electrical charge impressed upon the cable, and this charge for each signal must be got rid of before the succeeding signal can be sent into the cable.

It is, therefore, probable that a practical limit would soon be reached in the present form of transmitter, where the magnitude of this extra charge sent into the cable would become so great that there would be little advantage in further increasing the E.M.F.

The direct influence which increase of voltage produces on the amplitude of the record received as the frequency changes is shown in the graph (Fig. 7). These data were obtained over a submarine cable from New York City to Canso, Nova Scotia, of

$$R = 13{,}700 \text{ ohms.}$$

$$C = 231{\cdot}4 \cdot 10^{-6} \text{ farads.}$$

$$\text{Length} = 880{\cdot}6 \text{ nautical miles.}$$

The cable was used at the transmitting end simplex, without condensers, and at the receiving end the ordinary duplex arrangements were employed with 50 mf. condensers, and the cable recorder was adjusted once for all, and remained unchanged throughout the experiments. The receiving arrangements were not particularly sensitive, and the recorder was not readjusted as the frequency was increased.

It will be observed that, within the limits of these experiments, the amplitude of the excursions of the siphon increased with the voltage for any particular frequency of the dynamo, and as the frequency was increased, the voltage remaining

constant, the double amplitude of the siphon record gradually decreased. Theoretically, these lines are curved lines, but for observations up to 30 volts, with but one adjustment of the recorder, the right lines in this figure best represent the actual observations at each voltage.

It is seen that these lines converge towards a common vanishing point at somewhere about $n=7$, which means that,

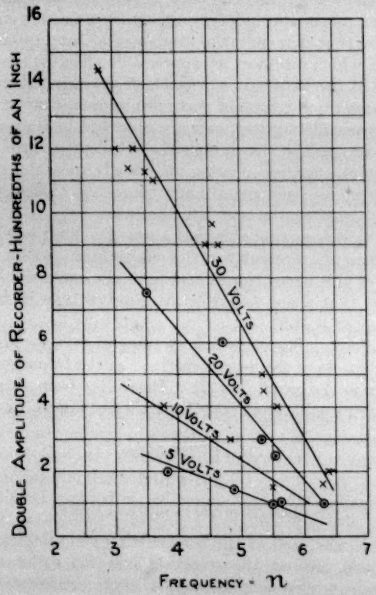

FIG. 7.—VARIATION OF DOUBLE AMPLITUDE OF SIPHON RECORD WITH CHANGE IN FREQUENCY, VOLTAGE CONSTANT.

with the limited sensibility of the particular recorder used (which may be considered as an ammeter if properly calibrated), we cannot expect by any increase of voltage to signal faster than about $n=7$ with this particular cable.

This graph has been made from data of experiments made

17 years ago, and since that date, due to the great improvement in sensibility of cable relays and amplifiers, the vanishing point at the present day for this cable would be moved to the right along the axis of absissa, to some point corresponding to n equal a considerably larger number than 7.

The above outline will be sufficient to show that in the case of an alternating current being impressed upon a cable, increase of voltage for signalling purposes should be considered from a new point of view, for we know that the power impressed in this case varies directly with the square of the voltage used in transmitting.

Maximum Voltage Along the Cable.

In (1) and (2) above, it is seen from the exponential factor that the maximum ordinates of the E.M.F. and current waves decay according to the logarithmic law, and that the rate of decay is dependent upon the capacity and the resistance of the cable per unit length, and also upon the frequency.

Assuming $n=5$, or 150 letters per minute, the instantaneous value of e or i, and also the maximum volts per volt at sending end have been computed for the Atlantic cable of Fig. 6, and are shown in Table I.

TABLE I.

Frequency $=n$.	Nautical miles.	e or i.	Maximum volst per volt at sending end.
5	0	1	1
"	135	0·5135	0·5924
"	270	0·1755	0·3510
"	405	0	0·2079
"	540	−0·0616	0·1231
"	675	−0·0630	0·0729
"	810	−0·0432	0·0432
"	945	−0·0222	0·0256
"	1,080	−0·0076	0·0152
"	1,215	0	0·0090
"	1,350	0·0027	0·0053
"	1,485	0·0027	0·0031
"	1,620	0·0019	0·0019

These relations are shown graphically in Fig. 8. It is seen from the table that the maximum voltage impressed upon the cable has decreased to 1/500 of its original value in one complete wave-length, and to 0·5924 of the original value in the first 135 nautical miles from the transmitting end of the cable.

At this speed of signalling a complete wave is 1,620 nautical miles in length; also we have

$$\lambda = 1{,}620 \text{ nautical miles}$$

$$T = \frac{1}{n} = \frac{1}{5} \text{ second.}$$

Total length of cable $= 1{\cdot}33\lambda$.

Rate of propagation of wave through the cable $= \frac{\lambda}{T} = 8{,}100$ nautical miles per second, or the time required for the wave to travel the length of the cable $= 0{\cdot}27$ of a second.

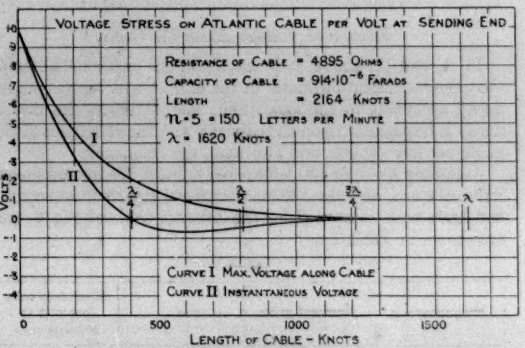

FIG. 8.—VOLTAGE STRESS ON TRANS-ATLANTIC CABLE FOR $n=5$, OR 150 LETTERS PER MINUTE.

The frequency n has a marked influence upon the rate at which the amplitude of the wave decreases as we proceed along the cable from the transmitting end. The distance at which this amplitude will have $\frac{1}{\varepsilon}$ of its original value is the reciprocal of the coefficient of x in the exponential term of (1), or

$$x' = \sqrt{\frac{2}{CR\omega}} = 258 \text{ nautical miles.}$$

or x' varies inversely with the square root of the frequency. This means that if we increase the speed of signalling from $n=4$ to $n=9$ the higher frequency wave will experience the same decrease in voltage in two-thirds of the distance from

the transmitting end. In other words, the higher the speed of signalling attained on any cable for a given transmitting voltage the safer the cable is from electrical strain. Increase in efficiency of the cable plant means increase in safety also.

With a battery, it is possible to subject every centimetre of the cable to the full voltage stress of the battery, as occurs whenever the usual insulation resistance test of the cable is made, or in case the transmitting condenser should be accidentally short-circuited.

With the alternating-current dynamo, however, it is impossible either by accident or by design to produce in the cable in the deep sea portion any voltage, even momentarily, of more than a small fraction of the voltage used at the transmitting end.

This suggests, as has been done before, that, by designing submarine cables with heavier insulation over comparatively short distances at the ends, we can employ higher voltages in signalling with no possibility of subjecting the cable to undue electrical strain at any point of its length.

Submarine cables before laying are at present subjected to a stress of several hundred volts in the cable tanks for a considerable period, and if there is a material advantage in speed in increasing the signalling voltage there would seem to be no good reason for hesitating in the slightest degree to increase the voltage moderately in actual traffic.

Receiving Conditions.

In a long cable there is no reactive influence upon an ammeter or voltmeter placed in any circuit at the transmitting end due to any variation in the form of apparatus inserted at the receiving end of the cable. Opening and closing the cable circuit produces no perceptible indication. The energy of the waves may be considered as leaving the transmitting end of the cable along the conductor path never to return.

We are, therefore, at liberty to consider the electrical conditions for best receiving, separately and independently of the conditions for sending, and then inquire how far these two sets of conditions can be made to harmonize with each other.

Fig. 9 shows one of the standard arrangements for operating on long cables in which the values of the condensers and the resistance of the recorder coil as shown have become more or less fixed quantities in cable practice. The condenser c is limited to this comparatively small value, as its main function

is to improve the definition of letters like "s" and "h," for there is no need to correct for letters like "a" and "n."

This lack of uniform resolving power for all letters, at present limits the free use of any value of condenser in this arm which would be the most suitable from electrical considerations to increase the value of an alternating current in the recorder coil.

In Fig. 9 the attenuated alternating current at the receiving end of the cable is divided at the point B of the duplex bridge, and again subdivided at A and C inversely as the vector-impedances of the paths.

In the recorder arm BC the ideal electrical conditions would be to so assign values to the receiving condenser c, and the inductance and resistance of the recorder coil, as to cause this

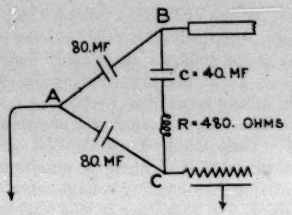

FIG. 9.—FORM OF DUPLEX CIRCUIT USED ON LONG CABLES.

arm of the bridge to be in resonance, entirely independently of the cable itself, the artificial line, or of any other arm of the bridge. In other words, we should so arrange matters if we could that if the arm BC is short-circuited on itself its natural frequency would be equal to the frequency of the current it is required to receive. In the abstract, there would be an infinite number of solutions for this arm, depending upon the stiffness function $\frac{L}{C}$ selected, each solution fulfilling the condition,

$$CL\omega^2=1.$$

The Cable Dynamo.

The alternator has been designed for the particular purpose of operating a cable, and possesses some novel features. Since we cannot reduce the number of poles to less than two, we must rely on a high magnetomotive force, and a minimum clearance

for the armature to produce, at such low frequencies, the voltage required.

In addition, it is especially desirable to generate an accurate sine wave of E.M.F. These objects have been satisfactorily obtained in the design of the machine used in these experiments.

The resistance of each of the field coils is nearly 1,000 ohms, and the inductance of the armature largely controls the current in the armature circuit, so that the machine can be short-circuited at full voltage without harm.

In starting and stopping the dynamo no form of switch is used, but, instead, the motor which operates the dynamo, starting from rest, gradually builds up the E.M.F. from zero, which gradually dies down to zero at stopping. In this manner there is never any possibility of a "break" in any metallic circuit connected with the transmitting apparatus.

Since the dynamo transmitter comprises the power plant for an ocean cable, it must, therefore, take its place on the instrument table, where the transmitter is directly geared to it. The armature is mounted on ball bearings, and runs with remarkable silence and smoothness under load.

The developments of wireless telegraphy have introduced alternating-current dynamos of considerable power of 100,000 and even 200,000 cycles per second; but for the present purpose we require an alternator to operate at from 4 to 10 cycles per second, and to develop 100 volts or more. This type of dynamo possesses some interest to the engineer, because it may be said to represent the extreme infra-red end of the dynamo spectrum, just as the high-frequency machines mentioned above represent the ultra-violet end. It would be strange indeed if the needs of telegraphy should be ultimately responsible for the development of both ends of this spectrum.

IV. *Summary.*

In the phenomenal development of the wireless art many instruments and methods have been imported from the much older art of wire telegraphy. In like manner may we not, to advantage, re-import some of the methods and instruments developed in wireless telegraphy into the art and science of cable telegraphy?

It has been truly said that "we begin to understand a phenomenon when we can measure it," and for this reason

alone the present plan, it is thought, will appeal to physicists and engineers.

It is seen that once we can employ a system of signalling in which we are dealing with one fundamental frequency only, and the circuit is never broken, the process of measuring the constants of the cable itself and the most suitable bridge arrangements for high efficiency is greatly simplified.

The removal of any form of arc or spark from the cable circuit has been shown to very materially increase the accuracy of balancing the duplex bridge, while the smooth type of waves employed in sending the signals has enabled this important object to be attained with an accuracy greater than has been possible before.

Using the low-resistance coil of a transformer or auto-transformer as the actual source of E.M.F. in the circuit comprising the cable itself, enables each end of the cable to be permanently connected to earth, thus providing a drainage path for earth currents of all kinds induced in the cable, and also adding to its safety.

The present forms of cable recorders, relays and amplifiers, when relieved from the necessity of recording consecutive impulses in the same direction, are liberated from some practical restrictions which have heretofore hampered them.

In the example shown the recorder coil of the gold-wire relay oscillates back and forth at a definite frequency, and the inertia of the moving parts is, therefore, less troublesome, and we are at liberty to increase the flux of the magnetic field, as well as reduce the control of the coil so as to allow it to vibrate more freely at its natural frequency.

No mention has been made of the possible application of this method of transmission to landlines, but it may be remarked that in the present Wheatstone automatic system there is great trouble caused by the induction of one line upon another, and if we could remove from the Wheatstone system the necessity of making and breaking the battery circuit at comparatively high voltage we could, at a single stroke, very largely reduce all such inductive effects.

The more usual method of developing a system of telegraphy is what might be called the "synthetical method"—that is, having certain fundamental principles which are proved to be correct, to endeavour to build up a system in practice which will possess as many as possible of these principles.

This method was employed in the earlier stages of these

experiments, but it was only after abandoning this course altogether, and starting with the ideal solution as outlined above, that it was possible to proceed without radical compromise. It is known, for instance, that certain characteristics are very desirable in any cable system, such as

(*a*) All signals should represent equal lengths of time;

(*b*) No two consecutive signals should be of the same sign;

(*c*) The resolving power or definition of each letter of the alphabet should be as nearly as possible the same;

(*d*) The total quantity of electricity measured in coulombs impressed upon the cable should be as nearly as possible equal to zero for any two consecutive signals.

It will be seen, however, that the ideal solution for power transmission automatically comprises each of these conditions, and, in addition, has other advantages in safety to the cable property from undue electrical strain throughout every centimetre of its length.

ABSTRACT.

1. The Paper proposes a new angle of view in the method of transmission of signals in the submarine telegraph cable, and describes some apparatus for operating on the general principles involved.

2. An ocean cable is considered as a power line, and starting with the standard form of circuit which would be used in case it were required to operate an electric motor through an ocean cable, experiments are described to determine the minimum possible variations required in such a circuit to permit the alternating current received to be interpreted in dots, dashes, and spaces of the present alphabet. The uninterrupted alternating current used in transmission is operated on synchronously by the ordinary transmitting tape, so as to alter the impedance of the transmitting circuit at the instants when the current is naturally zero. Dots, dashes and spaces are each sent by semi-waves of either sign, but of different amplitudes. The alternating current received may be read directly from the record made by a siphon recorder, or this current may be employed to operate a siphon Morse printer, by means of an adaptation of Muirhead's gold-wire relay, or a Heurtley magnifier and a local wire relay.

3. The voltage stress along an Atlantic cable when an alternator is employed is shown, and the transmitting impedance of such a cable is computed as the frequency varies.

4. A special form of cable dynamo to operate at frequencies from 4 to 10 was used in the experiments described.

5. The fundamental principle is developed of never metallically "breaking" the transmitter circuit, which permits of greater accuracy in balancing the duplex bridge.

DISCUSSION.

Mr. W. DUDDELL said the author and Dr. Crehore some 20 years ago were trying to get rid of the square corners of the waves due to the upper

harmonics. The difficulties now appeared to have been entirely surmounted. As pointed out in the Paper, if the current be broken the harmonics are again introduced, and he thought the method of avoiding this by exact synchronism of the transmitter and generator very ingenious.

Prof. S. P. THOMPSON thought the great merit of the method of working was that it reduced everything to the sine curve. Anything else than a sine curve was less economical financially and electrically. He admired the author's method of obtaining synchronism, and of making the alterations in amplitude exactly at the zero points.

Mr. A. CAMPBELL said that frequencies of 4 to 10 seemed to be usually employed. What were the limits of frequency practically possible on an Atlantic cable, for example? Was the limit set by the facility with which the signals could be read by the operator?

Dr. H. W. MALCOLM thought that by adopting the principle of never breaking the circuit the author was working on the right lines. The distortion in a long cable was so great that the wave-form of the signal received depended very little on the form transmitted, and so the wave-form could be chosen to produce the least disturbance at the transmitting end. For this the use of a series inductance was helpful. Another method was to use a commutator to shut off the recorder while the battery circuit was made or broken.

Prof. G. W. O. HOWE asked why the amplitude of the current which passed between signals could not be made zero. Was it an instrumental difficulty, or was there a theoretical advantage in having it large.

The AUTHOR, in reply, said it was desirable to have the amplitude large between signals, so as to depart as little as was practicable from its ideal sine wave. Moreover, the energy was utilised to feed the tape. With regard to frequency, 10 was very high from a cable standpoint, but he was hopeful that when the physicist had attended to the problem of the receiving instrument, much higher frequencies, say within the aural limit, would be possible.

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